## Chapter Planner

### Diagnostic Assessment
Quick Check, p. 85

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<tr>
<th>LESSON</th>
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<th>Pacing: 0.5 day</th>
<th>LESSON</th>
<th>Pacing: 2 days</th>
<th>EXTEND</th>
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<table>
<thead>
<tr>
<th>Title</th>
<th>Power and Radical Functions</th>
<th>Graphing Technology Lab: Behavior of Graphs</th>
<th>Polynomial Functions</th>
<th>Graphing Technology Lab: Hidden Behavior of Graphs</th>
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<tr>
<td>Objectives</td>
<td>■ Graph and analyze power functions.</td>
<td>■ Graph and analyze the behavior of polynomial functions.</td>
<td>■ Graph polynomial functions.</td>
<td>■ Model real-world data with polynomial functions.</td>
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<tr>
<td>Key Vocabulary</td>
<td>power function, monomial function, radical function, extraneous solutions</td>
<td>polynomial function, leading coefficient, leading-term test, turning point, quadratic form, repeated zero, multiplicity</td>
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<tr>
<td>NCTM Standards</td>
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<td>Multiple Representations</td>
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### Lesson Resources

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<tr>
<td>Study Notebook</td>
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</table>

### Resources for Every Lesson

- eStudent Edition
- Interactive Classroom
- eTeacher Edition
- Assessment

### Differentiated Instruction

- pp. 90, 91, 95
- pp. 100, 107

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All digital assets are Interactive Whiteboard ready.
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<th>LESSON 2-4</th>
<th>Pacing: 2 days</th>
<th>LESSON 2-5</th>
<th>Pacing: 2 days</th>
<th>LESSON 2-6</th>
<th>Pacing: 1 day</th>
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<tr>
<td><strong>The Remainder and Factor Theorems</strong></td>
<td><strong>Zeros of Polynomial Functions</strong></td>
<td><strong>Rational Functions</strong></td>
<td><strong>Nonlinear Inequalities</strong></td>
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<tr>
<td>▪ Divide polynomials using long division and synthetic division.</td>
<td>▪ Find real zeros of polynomial functions.</td>
<td>▪ Analyze and graph rational functions.</td>
<td>▪ Solve polynomial inequalities.</td>
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<tr>
<td>▪ Use the Remainder and Factor Theorems.</td>
<td>▪ Find complex zeros of polynomial functions.</td>
<td>▪ Solve rational equations.</td>
<td>▪ Solve rational inequalities.</td>
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<tr>
<td>synthetic division, depressed polynomial, synthetic substitution</td>
<td>Rational Zero Theorem, Descartes' Rule of Signs, Fundamental Theorem of Algebra, Linear Factorization Theorem, complex conjugates</td>
<td>rational function, asymptote, vertical asymptote, horizontal asymptote, oblique asymptote, holes</td>
<td>polynomial inequality, sign chart, rational inequality</td>
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<tr>
<td>p. 116</td>
<td>p. 128</td>
<td>p. 139</td>
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- [Worksheets](connectED.mcgraw-hill.com)
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- [Interactive Classroom](connectED.mcgraw-hill.com)
- [Assessment](connectED.mcgraw-hill.com)

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**Formative Assessment**
- Mid-Chapter Quiz, p. 118

**Suggested Pacing**

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<th>Instruction</th>
<th>Review &amp; Assess</th>
<th>Total</th>
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<tr>
<td>90-minute</td>
<td>6 days</td>
<td>1 day</td>
<td>7 days</td>
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Study Guide and Review, pp. 148–152
Practice Test, p. 153
Connect to AP Calculus pp. 154–155
### Assessment and Intervention


<table>
<thead>
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<th>Diagnosis</th>
<th>Prescription</th>
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<td><strong>Beginning Chapter 2</strong></td>
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<tr>
<td>Get Ready for the Chapter <strong>SE</strong></td>
<td>Response to Intervention <strong>TE</strong></td>
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<tr>
<td><strong>Beginning Every Lesson</strong></td>
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<tr>
<td>Then, Now, Why? <strong>SE</strong></td>
<td></td>
</tr>
<tr>
<td>5-Minute Check</td>
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</table>

| **During/After Every Lesson** |  |
| Guided Practice **SE**, every example |  |
| H.O.T. Problems **SE** |  |
| Spiral Review **SE** |  |
| Additional Examples **TE** |  |
| Watch Out! **TE** |  |
| Step 4, Assess **TE** |  |
| Self-Check Quizzes connectED.mcgraw-hill.com |  |

| **FORMATIVE ASSESSMENT** |  |
| Mid-Chapter Quiz **SE** |  |
| Assessment |  |

| **TIER 1 Intervention** Practice CRM connectED.mcgraw-hill.com |  |
| **TIER 2 Intervention** Differentiated Instruction **TE**; Study Guide and Intervention Masters **CRM** |  |

| **Before Chapter Test** |  |
| Chapter Study Guide and Review **SE** |  |
| Practice Test connectED.mcgraw-hill.com |  |
| Chapter Test connectED.mcgraw-hill.com |  |
| Vocabulary Review connectED.mcgraw-hill.com |  |
| Assessment |  |

| **TIER 1 Intervention** Practice CRM connectED.mcgraw-hill.com |  |
| **TIER 2 Intervention** Differentiated Instruction **TE**; Study Guide and Intervention Masters **CRM** |  |

| **SUMMATIVE ASSESSMENT** |  |
| **After Chapter 2** |  |
| Multiple-Choice Tests **CRM** |  |
| Free-Response Tests **CRM** |  |
| Vocabulary Test **CRM** |  |
| Extended Response Test **CRM** |  |
| Assessment |  |

| Study Guide and Intervention **CRM** connectED.mcgraw-hill.com |  |
| Study Guide and Intervention Masters **CRM** |  |
**Option 1  Reaching All Learners**

**Logical Learners** Have groups of students write and graph six functions, two for each of the following types:
\[ f(x) = x^n, \quad f(x) = x^{-n}, \quad \text{and} \quad f(x) = x^{p/n}, \]
where \( n \) and \( p \) are positive integers and \( \frac{p}{n} \) is in simplest form. Have students write each function and each graph on a separate index card. Groups trade cards and try to match each graph with its function.

**Visual/Spatial Learners** Have students use grid paper and colored pencils to help organize their work when performing synthetic division. For example, have students write \( 6x^3 - 25x^2 + 18x + 9 \) divided by \( x - 3 \) with color as shown. Then have them write on grid paper with the same colors.
Vertical Alignment

Before Chapter 2
- Evaluate polynomials and terms of polynomials.
- Graph quadratic functions.
- Solve quadratic equations and inequalities.
- Work with function notation and manipulate and evaluate functions.

Chapter 2
- Graph and analyze power, radical, polynomial, and rational functions.
- Divide polynomials using long division and synthetic division.
- Use the Remainder and Factor Theorems.
- Find all zeros of polynomial functions.
- Solve radical and rational equations.
- Solve polynomial and rational inequalities.

After Chapter 2
- Determine the domain and range of functions using graphs, tables, and symbols.
- Find the limit of a polynomial function, rational function, and radical function.
- Sketch the graph of a function.
- Solve real-world applications involving absolute extrema on a closed interval.

Lesson-by-Lesson Preview

2-1 Power and Radical Functions
Functions of the form \( f(x) = ax^n \), where \( a \) and \( n \) are constant real numbers, are power functions. A power function is also a type of monomial function. A monomial function is any function that can be written as \( f(x) = a \) or \( f(x) = ax^0 \), where \( a \) and \( n \) are nonzero constant real numbers.

If \( n \) is an even positive integer:

Even Degree

If \( n \) is an odd positive integer:

Odd Degree

Power functions of the form \( f(x) = x^n \) can be written as radical functions of the form \( f(x) = \sqrt[n]{x^p} \), where \( n \) and \( p \) are positive integers greater than 1 that have no common factors. Note that the domain may be restricted to nonnegative values.

If \( n \) is an even positive integer:

Even Degree

If \( n \) is an odd positive integer:

Odd Degree
2-2 Polynomial Functions
Monomial functions are the most basic polynomial functions. Finding the sums and differences of monomial functions creates other types of polynomial functions. The end behavior of a polynomial function \( f(x) = a_n x^n + \ldots + a_1 x + a_0 \) can be determined by the degree \( n \) of the polynomial and its leading coefficient \( a_n \).

A polynomial function \( f(x) = a_n x^n + \ldots + a_1 x + a_0 \) has at most \( n \) distinct real zeros and at most \( n - 1 \) turning points. The zeros can be found by factoring. The repeated zero \( c \) occurs when the factor \( (x - c) \) repeats itself. The number of times \( (x - c) \) occurs is the multiplicity of \( c \).

- odd multiplicity: the function’s graph crosses the \( x \)-axis at \( c \) and the value of \( f(x) \) changes signs at \( x = c \).
- even multiplicity: the function’s graph touches the \( x \)-axis at \( c \) and the value of \( f(x) \) does not change signs at \( x = c \).

2-3 The Remainder and Factor Theorems
An algorithm similar to the long division algorithm for integers can be used to divide polynomials. Division of polynomials can result in a zero remainder or in a nonzero remainder.

Synthetic division is a shortcut for dividing a polynomial by a linear factor of the form \( (x - c) \) using the coefficients of the dividend. So, a divisor such as \( (x - 2) \) would first be written as \( (x - (-2)) \).

\[
\begin{array}{c|ccc}
\multicolumn{1}{r|}{2} & 1 & 5 & -4 \\
\hline
& 1 & 3 & -10 \\
\end{array}
\]

Whether using long division or synthetic division, use zeros as placeholders for any missing term in the dividend and write the polynomial in standard form. If a polynomial function is divided by \( (x - c) \), then the remainder equals \( f(c) \) and \( (x - c) \) is a factor of the polynomial if and only if \( f(c) = 0 \).

2-4 Zeros of Polynomial Functions
Real zeros can be either rational or irrational. The Rational Zero Theorem, on p. 119, uses the leading coefficient and the constant term of a polynomial function with integer coefficients to determine all the possible rational zeros. Direct or synthetic substitution can be used to determine which of those possible zeros are actual zeros. Narrow the search for the actual zeros by determining the interval (upper and lower bounds) within which the zeros are located or by using Descartes’ Rule of Signs.

2-5 Rational Functions
The quotient of two polynomial functions is a rational function.

- Vertical asymptotes, if any, occur at real zeros (undefined values), if any, of the denominator of the function.
- \( y = 0 \) is a horizontal asymptote if the degree \( n \) of the numerator is less than the degree \( m \) of the denominator.
- No horizontal asymptotes occur if \( n > m \).
- If \( n = m \), there is a horizontal asymptote at the ratio of the leading coefficients of the numerator and denominator.
- If \( n = m + 1 \), where \( m > 0 \), the graph has an oblique asymptote.

2-6 Nonlinear Inequalities
The real zeros of a polynomial function divide the \( x \)-axis into intervals for which the values of \( f(x) \) are either entirely positive (the graph is above the \( x \)-axis) or entirely negative (the graph is below the \( x \)-axis). A polynomial inequality can be solved using a sign chart and its end behavior. A rational inequality can be solved by first writing the inequality in general form with a single rational expression on the left side and a 0 on the right and then creating a sign chart using its real zeros and undefined points.
# Chapter Project

**Olé Cocoa Café**

Students use what they learned about power, radical, polynomial and rational functions and inequalities to analyze aspects of a business.

- Have students think about selling cocoa to complete each statement.
  1. My café sells on average ____ cocoas each month at an average of $$\_\_\_\_$$ per beverage.
  2. I will sell ____ fewer cocoas for every $$\_\_\_\_$$ I raise the price.
  3. I want total sales to equal $$\_\_\_\_$$.
  4. For selling \( n \) cocoas, I will make \( p(n) = n^2 - ____ n \) hundreds of dollars in revenue. It costs me \( c(n) = n + ____ \) to make \( n \) cocoas.

- Have students write a function for their total sales after raising the price of each cocoa by \( x \) dollars. Have students determine how many dollars they need to raise the price of each cocoa so that the total amount of sales equals their number from statement 3 above.

- Have students use their answers to statement 4 to determine the minimum number of cocoas they need to sell in order to make a profit.

## Key Vocabulary

**Define:** An asymptote is a line or curve that a graph approaches.

**Example:** The line \( x = -2 \) is a vertical asymptote of the graph of \( f(x) = \frac{5}{(x + 2)^2} \).

**Ask:** Why can \( x \) never equal \(-2\) in \( f(x) = \frac{5}{(x + 2)^2} \)? The function is undefined for any value that makes the denominator equal to 0.

### Then | Now | Why?
---|---|---
**In Chapter 1,** you analyzed functions and their graphs and determined whether inverse functions existed. **In Chapter 2,** you will:

- Model real-world data with polynomial functions.
- Use the Remainder and Factor Theorems.
- Find real and complex zeros of polynomial functions.
- Analyze and graph rational functions.
- Solve polynomial and rational inequalities.

**ARCHITECTURE** Polynomial functions are often used when designing and building a new structure. Architects use functions to determine the weight and strength of the materials, analyze costs, estimate deterioration of materials, and determine the proper labor force.

**PREREAD** Scan the lessons of Chapter 2, and use what you already know about functions to make a prediction of the purpose of this chapter.

See students’ work.

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### Preread/Prewrite

**Anticipation Guide**

Encourage students to begin their study of the chapter by prereading each lesson. They should think about their background knowledge and make predictions about the content. Allow time for groups to discuss the reading and generate questions. Emphasize the text features such as section headings and Key Concept and Concept Summary Boxes. Students should also complete the first column of the Anticipation Guide as well as the chapter organizer from the Study Notebook to prepare for Chapter 2.
Essential Questions

- Why is mathematics used to model real-world situations? Sample answer: in order to study trends, make predictions, understand phenomena in nature

- When would a nonlinear function be used to model a real-world situation? Sample answer: When the relationship that is modeled has a rate of change that is not constant, and thus, is nonlinear.
1 Focus

Vertical Alignment

Before Lesson 2-1 Analyze parent functions and their families of graphs.

Lesson 2-1 Graph and analyze power functions. Graph and analyze radical functions, and solve radical equations.

After Lesson 2-1 Graph and analyze polynomial functions.

2 Teach

Preread/Prewrite

Study Notebook

Have students complete the What You’ll Learn section in the Study Notebook.

Scaffolding Questions

Have students read the Why? section of the lesson.

Ask:

- Can a steel cable have a diameter less than or equal to 0 inches? Explain. No, length is always positive.
- What will a graph of data comparing the cable diameter and the amount of weight it can support look like? Sample answer: a curve with no intercepts, increasing from left to right

Lesson 2-1 Resources

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<th>On Level (OL)</th>
<th>Beyond Level (BL)</th>
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<td>Differentiated Instruction</td>
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<td>Chapter Resource Masters</td>
<td>Study Guide and Intervention</td>
<td>Study Guide and Intervention</td>
<td>Practice</td>
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<td>Practice</td>
<td>Practice</td>
<td>Word Problem Practice</td>
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<td>Enrichment</td>
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<td>Other</td>
<td>Study Notebook</td>
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<td>5-Minute Check</td>
<td>5-Minute Check</td>
<td>5-Minute Check</td>
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</tbody>
</table>
Monomial functions with an even degree are also **even** in the sense that \( f(-x) = f(x) \). Likewise, monomial functions with an odd degree are also **odd**, or \( f(-x) = -f(x) \).

**Example 1: Analyze Monomial Functions**

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

- **a.** \( f(x) = \frac{1}{x^4} \)

  Evaluate the function for several \( x \)-values in its domain. Then use a smooth curve to connect each of these points to complete the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
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<td>8</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
<td>indf</td>
<td>40.5</td>
</tr>
</tbody>
</table>

  Domain: \((-\infty, \infty)\)  
  Range: \([0, \infty)\)  
  Intercept: 0  
  End behavior: \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = \infty \)  
  Continuity: continuous on \((-\infty, \infty)\)  
  Decreasing: \((-\infty, 0)\)  
  Increasing: \((0, \infty)\)

- **b.** \( f(x) = -x^7 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>128</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-128</td>
<td>-2187</td>
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</table>

  Domain: \((-\infty, \infty)\)  
  Range: \((-\infty, 0)\)  
  Intercept: 0  
  End behavior: \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \)  
  Continuity: continuous on \((-\infty, \infty)\)  
  Decreasing: \((-\infty, \infty)\)

**Guided Practice** 1A–B. See Chapter 2 Answer Appendix.

1A. \( f(x) = 3x^6 \)  
1B. \( f(x) = -\frac{2}{3}x^3 \)

Recall that \( f(x) = \frac{1}{x} \) or \( x^{-1} \) is undefined at \( x = 0 \). Similarly, \( f(x) = x^{-3} \) and \( f(x) = x^{-3} \) are undefined at \( x = 0 \). Because power functions can be undefined when \( n < 0 \), the graphs of these functions will contain discontinuities.

**Example 2: Functions with Negative Exponents**

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

- **a.** \( f(x) = 3x^{-2} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>undefined</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>0.75</td>
<td>3</td>
<td>undefined</td>
<td>0.75</td>
<td>0.3</td>
<td></td>
<td></td>
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</tbody>
</table>

  Domain: \((-\infty, 0) \cup (0, \infty)\)  
  Range: \((0, \infty)\)  
  Intercepts: none  
  End behavior: \( \lim_{x \to 0^+} f(x) = 0 \) and \( \lim_{x \to 0^-} f(x) = 0 \)  
  Continuity: infinite discontinuity at \( x = 0 \)  
  Increasing: \((-\infty, 0)\)  
  Decreasing: \((0, \infty)\)

**Additional Answers (Additional Example)**

1a.  
1b.
Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

**a.** $f(x) = 2x^{-4}$

- Domain: $(-\infty, 0) \cup (0, \infty)$
- Range: $(0, \infty)$
- No intercept; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite continuity at $x = 0$; increasing: $(-\infty, 0)$ and decreasing: $(0, \infty)$

**b.** $f(x) = 2x^{-3}$

- Domain: $(-\infty, 0) \cup (0, \infty)$
- Range: $(0, \infty)$
- No intercept; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$ and decreasing: $(0, \infty)$

**Guided Practice 2A** See Chapter 2 Answer Appendix.

**2A.** $f(x) = -\frac{3}{2}x^{-4}$

**2B.** $f(x) = 4x^{-3}$

### Rational Exponents

Recall that $x^\frac{a}{b}$ indicates the $b$th root of $x$, and $x^\frac{p}{n}$, where $\frac{p}{n}$ is in simplest form, indicates the $n$th root of $x^p$. If $n$ is an even integer, then the domain must be restricted to nonnegative values.

### Example 3 Rational Exponents

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

**a.** $f(x) = x^\frac{3}{2}$

- Domain: $[0, \infty)$
- Range: $[0, \infty)$
- $x$- and $y$-Intercepts: 0
- End behavior: $\lim_{x \to \infty} f(x) = \infty$
- Continuity: continuous on $[0, \infty)$
- Increasing: $(0, \infty)$

**b.** $f(x) = 6x^{-\frac{2}{3}}$

- Domain: $(-\infty, 0) \cup (0, \infty)$
- Range: $(0, \infty)$
- Intercepts: none
- End behavior: $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 0$
- Continuity: infinite discontinuity at $x = 0$
- Increasing: $(-\infty, 0)$

### Guided Practice 3A** See Chapter 2 Answer Appendix.

**3A.** $f(x) = 2x^\frac{3}{4}$

**3B.** $f(x) = 10x^\frac{5}{3}$

---

**Focus on Mathematical Content**

**Functions** A power function is any function of the form $f(x) = ax^n$, where $a$ and $n$ are nonzero constant real numbers. A monomial function is a power function in which $n$ is a positive integer. The graph of an even-degree monomial function is symmetric with respect to the $y$-axis. The graph of an odd-degree monomial function is symmetric with respect to the origin.

**Power Functions** A power function of the form $f(x) = ax^n$, where $n$ is a negative integer, contains a discontinuity. A power function of the form $f(x) = ax^\frac{p}{n}$, where $n$ is even and $\frac{p}{n}$ is in simplest form, has the domain restricted to nonnegative values.
**Biological Science** The following data represents the resting metabolic rate \( R \) in kilocalories per day for the mass \( m \) in kilograms of several selected animals.

<table>
<thead>
<tr>
<th>( m )</th>
<th>0.3</th>
<th>0.4</th>
<th>0.7</th>
<th>0.8</th>
<th>0.85</th>
<th>2.4</th>
<th>2.6</th>
<th>5.5</th>
<th>6.4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>28</td>
<td>35</td>
<td>54</td>
<td>66</td>
<td>46</td>
<td>135</td>
<td>143</td>
<td>331</td>
<td>320</td>
<td>292</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m )</th>
<th>7</th>
<th>7.9</th>
<th>8.41</th>
<th>8.5</th>
<th>13</th>
<th>29.3</th>
<th>29.8</th>
<th>83.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>265</td>
<td>327</td>
<td>346</td>
<td>363</td>
<td>520</td>
<td>895</td>
<td>839</td>
<td>1036</td>
</tr>
</tbody>
</table>

Source: American Journal of Physical Anthropology

a. Create a scatter plot of the data. The scatter plot appears to resemble the square root function, which is a power function. Therefore, test a power regression model.

b. Write a polynomial function to model the data set. Round each coefficient to the nearest thousandth, and state the correlation coefficient.

Using the PolyReg tool on a graphing calculator and rounding each coefficient to the nearest thousandth yields \( f(x) = 69.582x^{0.759} \). The correlation coefficient \( r \) for the data, 0.995, suggests that a power regression may accurately reflect the data.

We can graph the complete (unrounded) regression by sending it to the \( \text{Y=} \) menu. In the \( \text{VARS} \), Statistics, EQ, Graph this function and the scatter plot in the same viewing window. The function appears to fit the data reasonably well.

c. Use the equation to predict the resting metabolic rate for a 60-kilogram animal.

Use the \( \text{CALC} \) feature on the calculator to find \( f(60) \). The value of \( f(60) \) is about 1554, so the resting metabolic rate for a 60-kilogram animal is about 1554 kilocalories.

**Guided Practice**

4. **CARS** The table shows the braking distance in feet at several speeds in miles per hour for a specific car on a dry, well-paved roadway.

<table>
<thead>
<tr>
<th>Speed (mph)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (ft)</td>
<td>4.2</td>
<td>16.7</td>
<td>37.6</td>
<td>66.9</td>
<td>104.5</td>
<td>150.5</td>
<td>204.9</td>
</tr>
</tbody>
</table>

A. Create a scatter plot of the data. See Chapter 2 Answer Appendix.
B. Determine a power function to model the data. \( y = 0.042x^2 - 0.004x + 0.043 \)
C. Predict the braking distance of a car going 80 miles per hour. About 267.6 ft

**Radical Functions**

An expression with rational exponents can be written in radical form.

\[
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:} \\
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:} \\
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:} \\
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:} \\
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:} \\
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:} \\
\text{Exponential Form: } \quad x^{\frac{1}{n}} = \sqrt[n]{x} \\
\text{Radical Form:}
\]

Power functions with rational exponents represent the most basic of radical functions. A radical function is a function that can be written as \( f(x) = \sqrt[n]{x} \), where \( n \) and \( p \) are positive integers greater than 1 that have no common factors. Some examples of radical functions are shown below.

\[
f(x) = 3\sqrt[3]{x^3} \quad f(x) = -5\sqrt[4]{x^4} + 3x^2 - 1 \quad f(x) = \sqrt[4]{x} + 12 + \frac{3}{x^2} - 7
\]

a. Create a scatter plot of the data.

b. Determine a power function to model the data. \( y = 0.02x^{1.5} \)

c. Use the data to predict the mass of an African Golden cat with a length of 77 centimeters. About 14.1 kg
2 Radical Functions

Example 5 shows how to graph and analyze radical functions, describing the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. Example 6 shows how to solve radical equations, eliminating extraneous solutions.

Additional Example

5 Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

a. \( f(x) = 5\sqrt[3]{2x^3} \)
   \( D = [0, \infty); R = [0, \infty); \) intercept: 0
   \( \lim_{x \to \infty} f(x) = \infty; \) continuous on \([0, \infty); \) increasing: \((0, \infty)\)

b. \( f(x) = \frac{1}{2} \sqrt[3]{3x} - 4 \)
   \( D = (-\infty, \infty); R = (-\infty, \infty); \)
   \( x\)-intercept: \( \frac{4}{3} \) \( y\)-intercept: about \(-0.6598\)
   \( \lim_{x \to -\infty} f(x) = -\infty \) and
   \( \lim_{x \to \infty} f(x) = \infty; \) continuous for all real numbers; increasing: \((-\infty, \infty)\)

Example 5 Graph Radical Functions

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

a. \( f(x) = 2\sqrt{5x^3} \)

\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
x & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
f(x) & 2.09 & 5.03 & 8.62 & 10.46 & 12.00 \\
\hline
\end{array} \]

Domain and Range: \([0, \infty)\)
\( x\)- and \( y\)-Intercepts: 0
End behavior: \( \lim_{x \to \infty} f(x) = \infty \)
Continuity: continuous on \([0, \infty)\)
Increasing: \((0, \infty)\)

b. \( f(x) = \frac{1}{2} \sqrt[3]{6x} - 8 \)

\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -3 & -2 & -1 & 0 & 1 & 2 \\
\hline
f(x) & -0.48 & -0.46 & -0.42 & -0.38 & -0.29 & 0.33 \\
\hline
\end{array} \]

Domain and Range: \((-\infty, \infty)\)
\( x\)-Intercept: \( \frac{3}{2} \) \( y\)-Intercept: about \(-0.38\)
End behavior: \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \)
Continuity: continuous on \((-\infty, \infty)\)
Increasing: \((-\infty, \infty)\)

Guided Practice 5A–B. See margin.

5A. \( f(x) = -\sqrt[3]{12x^2} - 5 \)
5B. \( f(x) = \frac{1}{2} \sqrt[3]{2x^3} - 16 \)

It is important to understand the characteristics of the graphs of radical functions as well.

Key Concept Radical Functions

Let \( f \) be the radical function \( f(x) = \sqrt[n]{x} \) where \( n \) is a positive integer.

- \( \text{Domain and Range: } [0, \infty) \)
- \( x\)- and \( y\)-Intercepts: 0
- Continuity: continuous on \([0, \infty)\)
- Symmetry: none
- Increasing: \((0, \infty)\)
- \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \)

When \( n \) is even, the domain and range will have restrictions.

- \( \text{Domain and Range: } (-\infty, \infty) \)
- \( x\)- and \( y\)-Intercepts: 0
- Continuity: continuous on \((-\infty, \infty)\)
- Symmetry: origin
- Increasing: \((-\infty, \infty)\)
- \( \lim_{x \to \infty} f(x) = \infty \)

Interpersonal Learners Have students work in groups to compare solving radical equations to solving quadratic equations. Have groups write or give a short presentation about the differences and similarities in solution processes.
Like radical functions, a radical equation is any equation in which a variable is in the radicand. To solve a radical equation, first isolate the radical expression. Then raise each side of the equation to a power equal to the index of the radical to eliminate the radical.

Raising each side of an equation to a power sometimes produces extraneous solutions, or solutions that do not satisfy the original equation. It is important to check for extraneous solutions.

**Example 6: Solve Radical Equations**

Solve each equation.

a. \(2x = \sqrt{100 - 12x} - 2\)

\[
2x = \sqrt{100 - 12x} - 2 \\
2x + 2 = \sqrt{100 - 12x} \\
4x^2 + 8x + 4 = 100 - 12x \\
4x^2 + 20x - 96 = 0
\]

Factor.

-16 and 6 are solutions. 

\(x = -8\) or \(x = 3\)

CHECK \(x = -8\)

\(2(-8) = \sqrt{100 - 12(-8)} - 2\)

\(-16 = \sqrt{100 - 12(-8)} - 2\)

\(-16 = \sqrt{100} - 2\)

\(-16 = 10 - 2\)

\(-16 = 8\) \(\times\)

One solution checks and the other solution does not. Therefore, the solution is 3.

b. \(\sqrt{(x - 5)^2} + 14 = 50\)

\(
\sqrt{(x - 5)^2} = 36 \\
(x - 5)^2 = 36 \\
x - 5 = \pm 6 \\
x = 11\) or \(-1\)

A check of the solutions in the original equation confirms that the solutions are valid.

c. \(\sqrt{x - 2} = 5 - \sqrt{15 - x}\)

\[
x - 2 = 25 - 10\sqrt{15 - x} + (15 - x) \\
2x - 42 = -10\sqrt{15 - x} \\
4x^2 - 168x + 1764 = 100(15 - x) \\
4x^2 - 168x + 1764 = 1500 - 100x \\
4x^2 - 68x + 264 = 0 \\
4(x^2 - 17x + 66) = 0 \\
x - 6 = 0 \text{ or } x - 11 = 0 \\
x = 6 \text{ or } x = 11
\]

A check of the solutions in the original equation confirms that both solutions are valid.

**Guided Practice**

6A. \(3x = 3 + \sqrt{18x - 18}\) 1, 3

6B. \(\sqrt{4x^2 + 8} + 3 = 7\) 14

6C. \(\sqrt{x + 7} = 3 + \sqrt{2 - x}\) 2

**Additional Example**

**6.** Solve each equation.

a. \(2x = \sqrt{28x + 29} - 3 \quad -1, 5\)

b. \(12 = \sqrt{(x - 2)^2} + 8 \quad 10, -6\)

c. \(\sqrt{x + 1} = 1 + \sqrt{2x - 12} \quad 8\)

**Tips for New Teachers**

**Extraneous Solutions** Remind students that there is the possibility of extraneous solutions occurring as a result of squaring. Therefore, any possible solution must be checked.

**Additional Answers (Guided Practice)**

5A.

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**Kinesthetic Learners** Have students use a flow chart program or an Interactive Whiteboard to create a flow chart explaining how to solve a radical equation. Remind students that their flow charts should include a loop for the steps involved in isolating and eliminating the radicals. Then have students test their flow charts using equations from the Practice exercises.
3 Practice

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Examples 1 and 2)

1. \( f(x) = 5x^2 \)
2. \( g(x) = 8x^3 \)
3. \( h(x) = -x^3 \)
4. \( f(x) = -4x^4 \)
5. \( g(x) = \frac{3}{x^3} \)
6. \( f(x) = \frac{1}{x^4} \)
7. \( f(x) = -\frac{1}{2^2}x^2 \)
8. \( g(x) = -\frac{1}{3^2}x^6 \)
9. \( f(x) = 2x^{-4} \)
10. \( h(x) = -3x^{-7} \)
11. \( f(x) = -8x^{-5} \)
12. \( g(x) = 7x^{-2} \)
13. \( f(x) = -\frac{2}{3}x^{-9} \)
14. \( h(x) = \frac{3}{x^{-4}} \)
15. \( h(x) = \frac{3}{2x^{-3}} \)
16. \( f(x) = -\frac{7}{10}x^{-8} \)

17. GEOMETRY The volume of a sphere is given by \( V(r) = \frac{4}{3}\pi r^3 \), where \( r \) is the radius. (Example 1)

\[ f(x) = 1 \]
\[ 30a. f(x) = 2 \]
\[ 30b. f(x) = -3 \]
\[ 30c. f(x) = -4 \]

Describe the domain and range of the function. 

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Example 3)

18. \( f(x) = x^4 \)
19. \( f(x) = -6x^3 \)
20. \( g(x) = \frac{1}{3}x^{-\frac{3}{2}} \)
21. \( f(x) = 10x^{-3} \)
22. \( g(x) = -3x^{-\frac{3}{2}} \)
23. \( h(x) = \frac{3}{2}x^{-\frac{3}{2}} \)
24. \( f(x) = -\frac{1}{3}x^{-\frac{3}{2}} \)
25. \( f(x) = x^{-\frac{3}{2}} \)
26. \( h(x) = 2x^{-\frac{3}{2}} \)
27. \( h(x) = -4x^{-\frac{3}{2}} \)
28. \( h(x) = -5x^{-\frac{3}{2}} \)
29. \( h(x) = \frac{3}{2}x^{-\frac{3}{2}} \)
30. \( f(x) = 3\sqrt{x} + 3x \)
31. \( f(x) = -2\sqrt{x^2 + 8x} \)

Complete each step. 30a–31. See margin.

a. Create a scatter plot of the data.

b. Determine a power function to model the data.

c. Calculate the value of each model at \( x = 30 \). (Example 4)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>2</td>
<td>82</td>
</tr>
<tr>
<td>3</td>
<td>85</td>
<td>3</td>
<td>360</td>
</tr>
<tr>
<td>4</td>
<td>190</td>
<td>4</td>
<td>2000</td>
</tr>
<tr>
<td>5</td>
<td>370</td>
<td>5</td>
<td>7800</td>
</tr>
<tr>
<td>6</td>
<td>650</td>
<td>6</td>
<td>25,000</td>
</tr>
<tr>
<td>7</td>
<td>1000</td>
<td>7</td>
<td>60,000</td>
</tr>
<tr>
<td>8</td>
<td>1500</td>
<td>8</td>
<td>130,000</td>
</tr>
</tbody>
</table>

CLIFF DIVING In the sport of cliff diving, competitors perform three dives from a height of 28 meters. Judges award a score from 0 to 10 points based on degree of difficulty, take-off, positions, and water entrance. The table shows the speed of a diver at various distances in the dive. (Example 4)

a. Create a scatter plot of the data. See margin.

b. Determine a power function to model the data. \( f(x) = 4.42x^{0.5} \)

c. Use the function to predict the speed at which a diver would enter the water from a cliff dive of 30 meters.

\[ \text{about } 24.25 \text{ m/s} \]

33. WEATHER The wind chill temperature is the apparent temperature felt on exposed skin, taking into account the effect of the wind. The table shows the wind chill temperature produced at winds of various speeds when the actual temperature is 50°F. (Example 4)

\[
\begin{array}{|c|c|}
\hline
\text{Wind Speed (mph)} & \text{Wind Chill (°F)} \\
\hline
5 & 46.22 \\
10 & 46.04 \\
15 & 44.64 \\
20 & 43.60 \\
25 & 42.76 \\
30 & 42.04 \\
35 & 41.43 \\
40 & 40.88 \\
\hline
\end{array}
\]

a. Create a scatter plot of the data. See margin.

b. Determine a power function to model the data.

c. Use the function to predict the wind chill temperature when the wind speed is 65 miles per hour.

\[ \text{about } 39.54°F \]

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. (Example 5)

34. \( f(x) = 3\sqrt{x} + 3x \)
35. \( g(x) = -2\sqrt{x^2 + 8x} \)
36. \( f(x) = -\frac{3}{2}\sqrt{16x + 48} - 3 \)
37. \( h(x) = 4 + \sqrt{7} - 12 \)
38. \( g(x) = \sqrt{(1 - 4x)^2} - 16 \)
39. \( f(x) = -\sqrt{(25x - 7)^2} - 49 \)
40. \( h(x) = \frac{1}{2}\sqrt{27 - 2x} - 8 \)
41. \( g(x) = \sqrt{22} - x - 3 \sqrt{x - 3} \)

FLUID MECHANICS The velocity of the water flowing through a hose with a nozzle can be modeled using \( V(P) = 12.1\sqrt{P} \), where \( V \) is the velocity in feet per second and \( P \) is the pressure in pounds per square inch. (Example 5)

a. Graph the velocity through a nozzle as a function of pressure.

b. Describe the domain, range, end behavior, and continuity of the function and determine where it is increasing or decreasing.
Agricultural Science  The net energy $NE_{me}$ required to maintain the body weight of beef cattle, in megacalories (Mcal) per day, is estimated by the formula $NE_{me} = 0.077 \sqrt{m^2}$, where $m$ is the animal’s mass in kilograms. One megacalorie is equal to one million calories. (Example 6)

a. Find the net energy per day required to maintain a 400-kilogram steer. about 6.89 Mcal

b. If 0.96 megacalorie of energy is provided per pound of whole grain corn, how much corn does a 400-kilogram steer need to consume daily to maintain its body weight? about 7.18 lb

Solve each equation. (Example 6)  
44. no solution  
45. no solution  
46. $-4 = \sqrt{6 - 2x} + 3\sqrt{1 - 3x}$  
47. $0.5x = \sqrt{4 - 3x} + 2$  
48. $-3 = \sqrt{2x - x} - \sqrt{3x - 3}$  
49. $\sqrt{(2x - 5)^3} - 10 = 17$  
50. $7 + \sqrt{(-36 - 5x)^3} = 250$  
51. $x = 5 + \sqrt{x - 1}$  
52. $\sqrt{6x - 11} + 4 = \sqrt{12x + 1}$  
53. $\sqrt{4x - 40} - 20 = 0$  
54. $\sqrt{x + 2} - 1 = \sqrt{2x - 2x}$  
55. $x = \sqrt{1054 - 3x} - 11$  
56. $x = 4x - 1$  
57. $x = 4x - 1$

Chemistry  Boyle’s Law states that, at constant temperature, the pressure of a gas is inversely proportional to its volume. The results of an experiment to explore Boyle’s Law are shown.

<table>
<thead>
<tr>
<th>Volume (liters)</th>
<th>Pressure (atmospheres)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.85</td>
</tr>
<tr>
<td>1.5</td>
<td>2.41</td>
</tr>
<tr>
<td>2.0</td>
<td>1.75</td>
</tr>
<tr>
<td>2.5</td>
<td>1.46</td>
</tr>
<tr>
<td>3.0</td>
<td>1.21</td>
</tr>
<tr>
<td>3.5</td>
<td>1.02</td>
</tr>
<tr>
<td>4.0</td>
<td>0.92</td>
</tr>
</tbody>
</table>

a. Create a scatter plot of the data. See margin.

b. Determine a power function to model the pressure $P$ as a function of volume $v$. $P(v) = 3.62v^{-1.1}$

c. Based on the information provided in the problem statement, does the function you determined in part b make sense? Explain. See margin.

d. Use the model to predict the pressure of the gas if the volume is 3.25 liters. about 1.12 atmospheres

e. Use the model to predict the pressure of the gas if the volume is 6 liters. about 0.60 atmosphere

Without using a calculator, match each graph with the appropriate function.

Chemistry  The function $r = R_a(A)^3$ can be used to approximate the nuclear radius of an element based on its molecular mass, where $r$ is length of the radius in meters, $R_a$ is a constant (about $1.2 \times 10^{-13}$ meter), and $A$ is the molecular mass.

<table>
<thead>
<tr>
<th>Element</th>
<th>Molecular Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon</td>
<td>12.0</td>
</tr>
<tr>
<td>Helium</td>
<td>4.0</td>
</tr>
<tr>
<td>Iodine</td>
<td>126.9</td>
</tr>
<tr>
<td>Lead</td>
<td>207.2</td>
</tr>
<tr>
<td>Sodium</td>
<td>27.8</td>
</tr>
</tbody>
</table>

a. If the nuclear radius of sodium is about $3.412 \times 10^{-13}$ meter, what is its molecular mass? 23.0

b. The approximate nuclear radius of an element is $6.945 \times 10^{-13}$ meter. Identify the element. Iodine

c. The ratio of the molecular masses of two elements is 27.8. What is the ratio of their nuclear radii? 3.2

Additional Answers

32a. 

33a.

56. Yes; sample answer: The function follows the form $f(x) = ax^n$, where $n$ is a positive integer. In this case, $a = \frac{5}{b}$ and $n = 4a$.

57. Yes; sample answer: The function follows the form $f(x) = ax^n$, where $n$ is a positive integer. In this case, $a = -2a$ and $n = 4$.

58. No; sample answer: The function is not a power function because the variable is in the exponent.

59. Yes; sample answer: The function follows the form $f(x) = ax^n$, where $n$ is a positive integer. In this case, $a = \frac{7}{3}$ and $n = ab$.

60. Yes; sample answer: The function follows the form $f(x) = ax^n$, where $n$ is a positive integer. In this case, $a = \frac{1}{ab}$ and $n = 2b$.

61. No; sample answer: The function is not a monomial function because the exponent for $x$ is negative.

69a. Volume vs. Pressure

69c. Sample answer: Yes; the problem states that the volume and pressure are inversely proportional, and in the power function, the exponent of the volume variable is $-1$. 
48. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the average rates of change of power functions.

a. **GRAPHICAL** For power functions of the form \( f(x) = x^n \), graph a function with two values of \( n \) such that \( 0 < n < 1 \), \( n = 1 \), and two values of \( n \) such that \( n > 1 \).

b. **TABULAR** Copy and complete the table, using your graphs from part a to analyze the average rates of change of the functions as \( x \) approaches infinity.

Describe this rate as increasing, constant, or decreasing.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(x) )</th>
<th>Average Rate of Change as ( x \to \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; n &lt; 1 )</td>
<td>( f(x) = x^n )</td>
<td>decreases as ( x \to \infty )</td>
</tr>
<tr>
<td>( n = 1 )</td>
<td>( f(x) = x )</td>
<td>constant as ( x \to \infty )</td>
</tr>
<tr>
<td>( n &gt; 1 )</td>
<td>( f(x) = x^n )</td>
<td>increases as ( x \to \infty )</td>
</tr>
</tbody>
</table>

80a, c. See Chapter 2 Answer Appendix.

c. **VERBAL** Make a conjecture about the average rate of change of a power function as \( x \) approaches infinity for the intervals \( 0 < n < 1 \), \( n = 1 \), and \( n > 1 \).

**H.O.T. Problems** Use Higher-Order Thinking Skills

81. **CHALLENGE** Show that \( \sqrt[4]{8 + 27} = 2^{3/4} + 3^{2/3} \).

See Chapter 2.

82. **REASONING** Consider \( g(y) = y^2 \). **Answer Appendix**

a. Describe the value of \( y \) if \( y < 0 \). \( 0 < y < 1 \)

b. Describe the value of \( y \) if \( 0 < y < 1 \). \( 1 < y < 2 \)

c. Describe the value of \( y \) if \( y > 1 \). \( y > 2 \)

d. Write a conjecture about the relationship between the value of the base and the value of the power if the exponent is greater than or less than 1. Justify your answer. See Chapter 2 Answer Appendix.

83. **PREWRITE** Your senior project is to tutor an underclassman for four sessions on power and radical functions. Make a plan for writing that addresses purpose and audience, and has a controlling idea, logical sequence, and time frame for completion. See students’ work.

84. **REASONING** Given \( f(x) = \sqrt{x} \), where \( a \) and \( b \) are integers with no common factors, determine whether each statement is true or false. Explain.

a. If the value of \( a \) is even and the value of \( a \) is odd, then the function is undefined for \( x < 0 \).

b. If the value of \( a \) is even and the value of \( b \) is odd, then the function is undefined for \( x < 0 \).

c. If the value of \( a \) is 1, then the function is defined for all \( x \) \( a \), \( a \). See margin.

85. **REASONING** Consider \( f(x) = x^2 + 5 \). How would you expect the graph of the function to change as \( n \) increases if \( n \) is odd and greater than or equal to 3? See Chapter 2 Answer Appendix.

86. **WRITING IN MATH** Use words, graphs, tables, and equations to show the relationship between functions in exponential form and in radical form. See students’ work.

### Additional Answers

84a. True; sample answer: \( f(x) = \sqrt{x} \). If \( b \) is even and \( a \) is odd, then \( x \geq 0 \). An even root of a negative number is undefined.

84b. False; sample answer: \( f(x) = \sqrt{x} \). If \( b \) is odd and \( a \) is even, then \( f(x) \) is defined for all \( x \).
Spiral Review

87. **FINANCE** If you deposit $1000 in a savings account with an interest rate of r compounded annually, then the balance in the account after 3 years is given by \( B(t) = 1000(1 + r)^3 \), where \( r \) is written as a decimal. [Lesson 1-7]  
   a. Find a formula for the interest rate \( r \) required to achieve a balance of 8 in the account after 3 years.  
   b. What interest rate will yield a balance of $1100 after 3 years? 3.23%

Find \((f + g)(x), (f - g)(x), f(x)g(x), \text{ and } \left( \frac{f}{g} \right)(x)\) for each \( f(x) \) and \( g(x) \). State the domain of each new function. [Lesson 1-6]  
88. \( f(x) = x^2 - 2x \)  
   \( g(x) = x + 9 \)  
89. \( f(x) = \frac{x}{x + 1} \)  
   \( g(x) = x^2 - 1 \)  
90. \( f(x) = \frac{3}{x - 7} \)  
   \( g(x) = x^2 + 5x \)

Use the graph of \( f(x) \) to graph \( g(x) = |f(x)| \) and \( h(x) = f(|x|) \). [Lesson 1-5]  
91. \( f(x) = -4x + 2 \)  
92. \( f(x) = \sqrt{x + 3} - 6 \)  
93. \( f(x) = x^2 - 3x - 10 \)

**Crystal Ball**  
Ask students to write how they think today’s lesson on monomial functions will help them with tomorrow’s lesson on polynomial functions.

**Additional Answers**

84c. False; sample answer: \( f(x) \) can be written as \( f(x) = \sqrt{x^2} \). If \( b \) is odd, then \( f(x) \) is defined for all \( x \). If \( b \) is even, then \( x \geq 0 \) because even an root of a negative number is undefined.

88. \( (f + g)(x) = x^2 - x + 9 \)
   \( D = (-\infty, \infty) \); \( (f - g)(x) = x^2 - 3x - 9 \); \( D = (-\infty, \infty) \); \( (f \cdot g)(x) = x^3 + 7x^2 - 18x \);
   \( D = (-\infty, \infty) \); \( \left( \frac{f}{g} \right)(x) = \frac{x^2 - 2x}{x + 9} \);
   \( D = (-\infty, -9) \cup (-9, \infty) \)

89. \( (f + g)(x) = \frac{x^3 + x^2 - 1}{x + 1} \)
   \( D = (-\infty, -1) \cup (-1, \infty) \); \( (f - g)(x) = \frac{-x^3 - x^2 + 2x + 1}{x + 1} \);
   \( D = (-\infty, -1) \cup (-1, \infty) \);
   \( (f \cdot g)(x) = x^2 - x \);
   \( D = (-\infty, -1) \cup (-1, \infty) \); \( (f \left( \frac{1}{g} \right) \)(x) = \frac{x}{x^3 + x^2 - x - 1} \);
   \( D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \)

90. \( (f + g)(x) = \frac{x^3 - 2x^2 - 35x + 3}{x - 7} \);
   \( D = (-\infty, 7) \cup (7, \infty) \);
   \( (f - g)(x) = \frac{x^3 - 2x^2 - 35x}{x - 7} \);
   \( D = (-\infty, 7) \cup (7, \infty) \);
   \( (f \cdot g)(x) = \frac{3x^2 + 15x}{x - 7} \);
   \( D = (-\infty, 7) \cup (7, \infty) \);
   \( \left( \frac{f}{g} \right)(x) = \frac{3}{x^3 - 2x^2 - 35x} \);
   \( D = (-\infty, -5) \cup (-5, 0) \cup (0, 7) \cup (7, \infty) \)

**Skills Review for Standardized Tests**

97. SAT/ACT  
   If \( m \) and \( n \) are both positive, then which of the following is equivalent to \( \frac{2m\sqrt{81} + \sqrt{2}}{m^2} \)?  
   A. \( 3m\sqrt{2} \)  
   B. \( 6m\sqrt{2} \)  
   C. \( 4\sqrt{2} \)

98. REVIEW  
   If \( f(x, y) = x^2y^3 \) and \( f(a, b) = 10 \), what is the value of \( f(2a, 2b) \)?  
   F. 50  
   G. 100  
   H. 160

99. REVIEW  
   The number of minutes \( m \) it takes \( c \) children to eat \( p \) pieces of pizza varies directly as the number of pieces of pizza and inversely as the number of children. If it takes 5 children 30 minutes to eat 10 pieces of pizza, how many minutes should it take 15 children to eat 50 pieces of pizza?  
   C. 50  
   D. 60

100. If \( \sqrt{5m + 2} = 3 \), then \( m = ? \)  
    F. 3  
    G. 4  
    H. 5  
    J. 6

**Extension**  
Ask students to indicate a set of values for \( a \) and \( b \) in \( a = \sqrt{x + b} \) for which the result is always at least one real solution. When both \( a \) and \( b \) are positive integers or when \( a \) is a positive integer and \( b \) is a negative integer, the result is always at least one real solution.
1 Focus

Objective Graph and analyze the behavior of polynomial functions.

Teaching Tip
If students find that not enough of the graph is shown in the standard viewing window, they can press [Zoom] 3 while in the graphing window to zoom out, allowing more of the graph to be shown.

2 Teach

Working in Cooperative Groups
Pair students, mixing abilities. Have students work through the Activity parts a–c.

Ask:
- How is the equation in part a similar to the equation in part b? They both have the terms $6x^2 - 4x + 2$.
- How is the graph of the equation in part a the same as the graph of the equation in part b? They both pass through (0, 2).
- Look at the equation in part c. Would you expect its graph to pass through (0, 2) also? Explain. Yes; its constant term is also 2.

Have students work through the Analyze the Results Exercises 1–4.

Practice Have students complete Exercises 5–10.

3 Assess

Formative Assessment
Use Exercise 9 to assess whether students can graph a polynomial function and describe its end behavior.

From Concrete to Abstract
Ask students to predict what the end behavior of the graph of each equation in the Activity would be if the leading coefficients changed signs. Then have students graph both forms of each equation to check their predictions.

Additional Answers
1. Sample answer: The first terms are all different. The first term of $f(x)$ has a positive coefficient and the first term of $g(x)$ has a negative coefficient. $h(x)$ has another term in front, $-x^3$.
2. Sample answer: From $f(x)$ to $g(x)$, the end behavior appears reversed. $h(x)$ is different from the others in that as $x$ approaches both positive and negative infinity, $h(x)$ approaches negative infinity.
1 **Graph Polynomial Functions**  In Lesson 2-1, you learned about the basic characteristics of monomial functions. Monomial functions are the most basic polynomial functions. The sums and differences of monomial functions form other types of **polynomial functions**.

Let \( n \) be a nonnegative integer and let \( a_n, a_{n-1}, \ldots, a_1, a_0 \) be real numbers with \( a_n \neq 0 \). Then the function given by

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

is called a **polynomial function of degree** \( n \). The **leading coefficient** of a polynomial function is the coefficient of the variable with the greatest exponent. The leading coefficient of \( f(x) \) is \( a_n \).

You are already familiar with the following polynomial functions.

<table>
<thead>
<tr>
<th>Constant Functions</th>
<th>Linear Functions</th>
<th>Quadratic Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = c, \ c \neq 0 )</td>
<td>( f(x) = ax + c )</td>
<td>( f(x) = ax^2 + bx + c )</td>
</tr>
</tbody>
</table>

The zero function is a constant function with no degree. The graphs of polynomial functions share certain characteristics.

<table>
<thead>
<tr>
<th>Graphs of Polynomial Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Example</strong></td>
</tr>
<tr>
<td><img src="connectED.mcgraw-hill.com" alt="Graphs" /></td>
</tr>
</tbody>
</table>

Polynomial functions are defined and continuous for all real numbers and have smooth, rounded turns. Graphs of polynomial functions do not have breaks, holes, gaps, or sharp corners.

---

**Lesson 2-2 Resources**

<table>
<thead>
<tr>
<th>Resource</th>
<th>Approaching Level AL</th>
<th>On Level OL</th>
<th>Beyond Level EL</th>
<th>English Learners ELL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher Edition</td>
<td>Differentiated Instruction</td>
<td>Differentiated Instruction</td>
<td>Differentiated Instruction</td>
<td>Differentiated Instruction</td>
</tr>
<tr>
<td>Chapter Resource Masters</td>
<td>Study Guide and Intervention</td>
<td>Study Guide and Intervention</td>
<td>Practice</td>
<td>Study Guide and Intervention</td>
</tr>
<tr>
<td></td>
<td>Practice</td>
<td>Practice</td>
<td>Word Problem Practice</td>
<td>Practice</td>
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<tr>
<td></td>
<td>Word Problem Practice</td>
<td>Enrichment</td>
<td>Word Problem Practice</td>
<td>Word Problem Practice</td>
</tr>
<tr>
<td>Other</td>
<td>Study Notebook</td>
<td>Study Notebook</td>
<td>Study Notebook</td>
<td>Study Notebook</td>
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<tr>
<td></td>
<td>5-Minute Check</td>
<td>5-Minute Check</td>
<td>5-Minute Check</td>
<td>5-Minute Check</td>
</tr>
</tbody>
</table>

(continued on the next page)
1 Graph Polynomial Functions

Example 1 shows how to graph transformations of monomial functions. Example 2 shows how to apply the leading term test to a polynomial function written in standard form to determine the end behavior of the graph of the function. Examples 3–5 show how to find zeros of a polynomial function. Example 6 shows how to use the leading term test to a polynomial function.

Formative Assessment

Use the Guided Practice exercises after each example to determine students’ understanding of concepts.

Additional Example

1 Graph each function.
   a. \( f(x) = (x - 3)^5 \)

   b. \( f(x) = x^6 - 1 \)

Recall that the graphs of even-degree, non-constant monomial functions resemble the graph of \( f(x) = x^2 \), while the graphs of odd-degree monomial functions resemble the graph of \( f(x) = x^3 \). You can use the basic shapes and characteristics of even- and odd-degree monomial functions and what you learned in Lesson 1–5 about transformations to transform graphs of monomial functions.

Example 1 Graph Transformations of Monomial Functions

Graph each function.
   a. \( f(x) = (x - 2)^5 \)

   b. \( g(x) = -x^4 + 1 \)

This is an odd-degree function, so its graph is similar to the graph of \( y = x^3 \). The graph of \( f(x) = (x - 2)^5 \) is the graph of \( y = x^3 \) translated 2 units to the right.

This is an even-degree function, so its graph is similar to the graph of \( y = x^2 \). The graph of \( g(x) = -x^4 + 1 \) is the graph of \( y = x^4 \) reflected in the \( x \)-axis and translated 1 unit up.

In Lesson 1–3, you learned that the end behavior of a function describes how the function behaves, rising or falling, at either end of its graph. As \( x \to -\infty \) and \( x \to \infty \), the end behavior of any polynomial function is determined by its leading term. The leading term test uses the power and coefficient of this term to determine polynomial end behavior.

Key Concept Leading Term Test for Polynomial End Behavior

The end behavior of any non-constant polynomial function \( f(x) = a_nx^n + \cdots + a_1x + a_0 \) can be described in one of the following four ways, as determined by the degree \( n \) of the polynomial and its leading coefficient \( a_n \).

- \( n \) odd, \( a_n \) positive
- \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \)
- \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \)

- \( n \) odd, \( a_n \) negative
- \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \)
- \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \)

- \( n \) even, \( a_n \) positive
- \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = \infty \)
- \( \lim_{x \to \infty} f(x) = -\infty \) and \( \lim_{x \to -\infty} f(x) = -\infty \)

- \( n \) even, \( a_n \) negative
- \( \lim_{x \to \infty} f(x) = -\infty \) and \( \lim_{x \to -\infty} f(x) = -\infty \)
- \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = \infty \)

Additional Answers (Guided Practice)

1A. \( f(x) = 4 - x^2 \)

1B. \( g(x) = (x + 7)^4 \)
### Example 2: Apply the Leading Term Test

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test.

**a.** \( f(x) = 3x^4 - 5x^2 - 1 \)

The degree is 4, and the leading coefficient is 3. Because the degree is even and the leading coefficient is positive, \( \lim_{{x \to \infty}} f(x) = \infty \) and \( \lim_{{x \to -\infty}} f(x) = \infty \).

**b.** \( g(x) = -3x^2 - 2x^3 + 4x^4 \)

Write in standard form as \( g(x) = -2x^2 + 4x^4 - 3x^3 \). The degree is 7, and the leading coefficient is \(-2\). Because the degree is odd and the leading coefficient is negative, \( \lim_{{x \to -\infty}} f(x) = -\infty \) and \( \lim_{{x \to \infty}} f(x) = -\infty \).

**c.** \( h(x) = x^3 - 2x^2 \)

The degree is 3, and the leading coefficient is 1. Because the degree is odd and the leading coefficient is positive, \( \lim_{{x \to \infty}} f(x) = \infty \) and \( \lim_{{x \to -\infty}} f(x) = -\infty \).

### Guided Practice

**2A.** \( g(x) = 4x^5 - 8x^3 + 20 \)

**2B.** \( h(x) = -2x^6 + 11x^4 + 2x^2 \)

Consider the shapes of a few typical third-degree polynomial or cubic functions and fourth-degree polynomial or quartic functions shown.

Turning points indicate where the graph of a function changes from increasing to decreasing, and vice versa. Maxima and minima are also located at turning points. Notice that cubic functions have at most 2 turning points, and quartic functions have at most 3 turning points. These observations can be generalized as follows and shown to be true for any polynomial function.

### Teach with Tech

**Video Recording** Have students create a video explaining how to use the leading term test to determine the end behavior of a polynomial’s graph. Post the video to a class Web site so students can use it as an additional reference outside of class.
State the number of possible real zeros and turning points of \( f(x) = x^3 + 5x^2 + 4x \). Then determine all of the real zeros by factoring. The degree is 3, so \( f \) has at most 3 distinct real zeros and at most 2 turning points. \( f(x) = x^3 + 5x^2 + 4x = x(x + 1)(x + 4) \), so \( f \) has three zeros, 0, -1, and -4.

**Focus on Mathematical Content**

**Polynomial Functions**

Polynomial functions are formed by adding or subtracting monomial functions and constants.

- **Degree:** \( n \)
- **Maximum number of turning points:** \( n - 1 \)
- **At a zero of odd multiplicity:** The graph crosses the \( x \)-axis.
- **At a zero of even multiplicity:** The graph touches the \( x \)-axis.
- **Between zeros:** The graph is either above or below the \( x \)-axis.
- **The end behavior:** The graph is determined by its leading term using the leading term test.

**Study Tip**

Look Back: Recall from Lesson 1-2 that the \( x \)-intercepts of the graph of a function are also called the zeros of a function. The solutions of the corresponding equation are called the roots of the equation.

**Key Concept**

**Zeros and Turning Points of Polynomial Functions**

A polynomial function \( f \) of degree \( n \geq 1 \) has at most \( n \) distinct real zeros and at most \( n - 1 \) turning points.

| Example | Let \( f(x) = 3x^4 - 10x^3 - 15x^2 \). Then \( f \) has at most 6 distinct real zeros and at most 5 turning points. The graph of \( f \) suggests that the function has 3 real zeros and 3 turning points. |

Recall that if \( f \) is a polynomial function and \( c \) is an \( x \)-intercept of the graph of \( f \), then it is equivalent to say that:

- \( c \) is a zero of \( f \),
- \( x = c \) is a solution of the equation \( f(x) = 0 \), and
- \( (x - c) \) is a factor of the polynomial \( f(x) \).

You can find the zeros of some polynomial functions using the same factoring techniques you used to solve quadratic equations.

**Example 3**

**Zeros of a Polynomial Function**

State the number of possible real zeros and turning points of \( f(x) = x^3 - 5x^2 + 6x \). Then determine all of the real zeros by factoring.

The degree of the function is 3, so \( f \) has at most 3 distinct real zeros and at most \( 3 - 1 \) or 2 turning points. To find the real zeros, solve the related equation \( f(x) = 0 \) by factoring:

\[
x^3 - 5x^2 + 6x = 0
\]

Set \( f(x) \) equal to 0.

\[
x(x^2 - 5x + 6) = 0
\]

Factor the greatest common factor, \( x \).

\[
x(x - 2)(x - 3) = 0
\]

Factor completely.

So, \( f \) has three distinct real zeros, 0, 2, and 3. This is consistent with a cubic function having at most 3 distinct real zeros.

**CHECK**

You can use a graphing calculator to graph \( f(x) = x^3 - 5x^2 + 6x \) and confirm these zeros.

Additionally, you can see that the graph has 2 turning points, which is consistent with cubic functions having at most 2 turning points.

**Guided Practice**

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

3A. \( f(x) = x^3 - 6x^2 - 27x \)

4 zeros and 3 turning points; \( \pm \sqrt{3} \) and \( \pm \sqrt{3} \)

3B. \( f(x) = x^4 - 8x^2 + 15 \)

3 zeros and 2 turning points; 0, 9, and \(-3\)

In some cases, a polynomial function can be factored using quadratic techniques if it has **quadratic form**.

**Key Concept**

**Quadratic Form**

**Words**

A polynomial expression in \( x \) is in **quadratic form** if it is written as \( ax^2 + bx + c \) for any numbers \( a, b, \) and \( c, a \neq 0 \), where \( u \) is some expression in \( x \).

**Symbols**

\( x^4 - 5x^2 - 14 \) is in quadratic form because the expression can be written as \( (x^2)^2 - 5(x^2) - 14 \). If \( u = x^2 \), then the expression becomes \( u^2 - 5u - 14 \).

**Differentiated Instruction**

**Interpersonal Learners**

Have students work together in groups to sketch the graphs of polynomial functions having a particular degree and number of real roots, for example, degree 3 and 3 real roots or degree 3 and only 1 real root. Then have students experiment with coefficients in the general form of a polynomial in order to find functions with graphs that resemble their sketches.
Example 4: Zeros of a Polynomial Function in Quadratic Form

State the number of possible real zeros and turning points for \( g(x) = x^4 - 3x^2 - 4 \). Then determine all of the real zeros by factoring.

The degree of the function is 4, so \( g \) has at most 4 distinct real zeros and at most 4 – 1 or 3 turning points. This function is in quadratic form because \( x^4 - 3x^2 - 4 = (x^2)^2 - 3(x^2) - 4 \). Let \( u = x^2 \).

\[
\begin{align*}
(x^2)^2 - 3(x^2) - 4 &= 0 & \text{Set } g(x) \text{ equal to } 0. \\
u^2 - 3u - 4 &= 0 & \text{Substitute } u \text{ for } x^2. \\
(u + 1)(u - 4) &= 0 & \text{Factor the quadratic expression.} \\
(x^2 + 1)(x^2 - 4) &= 0 & \text{Substitute } x^2 \text{ for } u. \\
(x^2 + 1)(x - 2)(x + 2) &= 0 & \text{Factor completely.}
\end{align*}
\]

\( x^2 + 1 = 0 \) or \( x + 2 = 0 \) or \( x - 2 = 0 \) \text{ Zero Product Property} \[ x = \pm \sqrt{-1} \quad x = -2 \quad x = 2 \] \text{ Solve for } x.

Because \( \pm \sqrt{-1} \) are not real zeros, \( g \) has two distinct real zeros, \(-2 \) and \( 2 \). This is consistent with a quartic function. The graph of \( g(x) = x^4 - 3x^2 - 4 \) in Figure 2.2.1 confirms this. Notice that there are 3 turning points, which is also consistent with a quartic function.

Guided Practice

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

4A. \( g(x) = x^4 - 9x^2 + 18 \) \text{ 4B. } 5 \text{ real zeros and 4 turning points; } 0, \pm \sqrt{6} 

4A. 4 real zeros and 3 turning points; \( \pm \sqrt{3}, \pm \sqrt{6} \)

If a factor \( (x - c) \) occurs more than once in the completely factored form of \( f(x) \), then its related zero \( c \) is called a repeat zero. When the zero occurs an even number of times, the graph will be tangent to the \( x \)-axis at that point. When the zero occurs an odd number of times, the graph will cross the \( x \)-axis at that point. A graph is tangent to an axis when it touches the axis at that point, but does not cross it.

Example 5: Polynomial Function with Repeated Zeros

State the number of possible real zeros and turning points of \( h(x) = -x^4 - x^3 + 2x^2 \). Then determine all of the real zeros by factoring.

The degree of the function is 4, so \( h \) has at most 4 distinct real zeros and at most 4 – 1 or 3 turning points. Find the real zeros.

\[
\begin{align*}
-x^4 - x^3 + 2x^2 &= 0 & \text{Set } h(x) \text{ equal to } 0. \\
-x^2(x^2 + x - 2) &= 0 & \text{Factor the greatest common factor, } -x^2. \\
-x^2(x - 1)(x + 2) &= 0 & \text{Factor completely.}
\end{align*}
\]

The expression above has 4 factors, but solving for \( x \) yields only 3 distinct real zeros, 0, 1, and \(-2 \). Of the zeros, 0 occurs twice.

The graph of \( h(x) = -x^4 - x^3 + 2x^2 \) shown in Figure 2.2.2 confirms these zeros and shows that \( h \) has three turning points. Notice that at \( x = 1 \) and \( x = -2 \), the graph crosses the \( x \)-axis, but at \( x = 0 \), the graph is tangent to the \( x \)-axis.

Guided Practice

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring.

5A. \( g(x) = -2x^3 - 4x^2 + 16x \) \text{ 5B. } 5 \text{ real zeros and 4 turning points; } 0, 3 

5B. 3 real zeros and 2 turning points; \(-4, 0, 2 \)

Additional Examples

4 State the number of possible real zeros and turning points for \( h(x) = x^4 - 4x^2 + 3 \). Then determine all of the real zeros by factoring. The degree is 4, so \( h \) has at most 4 distinct real zeros and at most 3 turning points.

\( h(x) = x^4 - 4x^2 + 3 = (x^2 - 3)(x - 1)(x + 1) \), so \( h \) has four distinct real zeros, \( \pm \sqrt{3}, -1, \) and 1.

5 State the number of possible real zeros and turning points of \( h(x) = x^4 + 5x^3 + 6x^2 \). Then determine all of the real zeros by factoring. The degree is 4, so \( h \) has at most 4 distinct real zeros and at most 3 turning points.

\( h(x) = x^4 + 5x^3 + 6x^2 = x^2(x + 2)(x + 3) \), so \( h \) has three zeros, 0, \(-2 \), and \(-3 \). Of the zeros, 0 is repeated.

Tips for New Teachers

Zeros: Emphasize to students that they can confirm the zeros (where the graph crosses the \( x \)-axis) and the number of turning points by using a graphing calculator to graph the polynomial function.

Tangents: Explain to students that the graph of a polynomial function can be tangent to the \( x \)-axis at a specific point and intersect the \( x \)-axis at a different point, as shown in Figure 2.2.2.
For \( f(x) = x(3x + 1)(x - 2)^2 \),
(a) apply the leading-term test,
(b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function.

a. The degree is 4 and leading coefficient is 3, so \( \lim_{x \to \pm \infty} f(x) = \infty \).

b. The zeros are \( x = 0, x = -\frac{1}{3}, x = 2 \). The zero at \( x = 2 \) has multiplicity 2.

c. Sample answer: \((-1, 18), (0.1, -0.3087), (1, 4), (3, 30)\)

d. Plot the points you found (Figure 2.2.3).

You now have several tests and tools to aid you in graphing polynomial functions.

For \( f(x) = x(2x + 3)(x - 1)^2 \), (a) apply the leading-term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function.

a. The product \( x(2x + 3)(x - 1)^2 \) has a leading term of \( x(2x)^2 \) or \( 2x^4 \), so \( f \) has degree 4 and leading coefficient 2. Because the degree is even and the leading coefficient is positive, \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = \infty \).

b. The distinct real zeros are 0, -1.5, and 1. The zero at 1 has multiplicity 2.

c. Choose \( x \)-values that fall in the intervals determined by the zeros of the function.

<table>
<thead>
<tr>
<th>Interval</th>
<th>( x )-value in Interval</th>
<th>( f(x) )</th>
<th>( x, f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -1.5))</td>
<td>-2</td>
<td>f(-2) = 18</td>
<td>(-2, 18)</td>
</tr>
<tr>
<td>((-1.5, 0))</td>
<td>-1</td>
<td>f(-1) = -4</td>
<td>(-1, -4)</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>0.5</td>
<td>f(0.5) = 0.5</td>
<td>(0.5, 0.5)</td>
</tr>
<tr>
<td>(1, \infty)</td>
<td>1.5</td>
<td>f(1.5) = 2.25</td>
<td>(1.5, 2.25)</td>
</tr>
</tbody>
</table>

For each function, (a) apply the leading-term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function. 6A–B. See Chapter 2 Answer Appendix.

6A. \( f(x) = -2x(x - 4)(3x - 1)^3 \)  
6B. \( h(x) = -x^3 + 2x^2 + 8x \)
2 Model Data You can use a graphing calculator to model data that exhibit linear, quadratic, cubic, and quartic behavior by first examining the number of turning points suggested by a scatter plot of the data.

3 Real-World Example 7 Model Data Using Polynomial Functions

SAVINGS Refer to the beginning of the lesson. The average personal savings as a percent of disposable income in the United States is given in the table.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>% Savings</td>
<td>9.4</td>
<td>10.0</td>
<td>7.0</td>
<td>4.6</td>
<td>2.3</td>
<td>1.8</td>
<td>2.1</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Source: U.S. Department of Commerce

a. Create a scatter plot of the data and determine the type of polynomial function that could be used to represent the data.

b. Write a polynomial function to model the data set. Round each coefficient to the nearest thousandth, and state the correlation coefficient.

Using the QuadrReg tool on a graphing calculator and rounding each coefficient to the nearest thousandth yields \( f(x) = -0.009x^2 + 0.033x + 9.744 \). The correlation coefficient \( r^2 \) for the data is 0.96, which is close to 1, so the model is a good fit.

c. Use the model to estimate the percent savings in 1993.

Because 1993 is 23 years after 1970, use the \( f(23) \) feature on a calculator to find \( f(23) \). As shown in Figure 2.2.5, the regression equation will be entered into \( Y_1 \). Graph this function and the scatter plot in the same viewing window. The function appears to fit the data reasonably well.

The value of \( f(23) \) is 5.94, so the percent savings in 1993 was about 5.94%.

d. Use the model to determine the approximate year in which the percent savings reached 6.5%.

Graph the line \( y = 6.5 \) for \( Y_2 \). Then use \( 5: \) intersect on the \( \text{CALC} \) menu to find the point of intersection of \( y = 6.5 \) with \( f(x) \). The intersection occurs when \( x = 21 \), so the approximate year in which the percent savings reached 6.5% was about 1970 + 21 or 1991.

Guided Practice

7. POPULATION The median age of the U.S. population by year predicted through 2080 is shown.

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
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<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Median Age</td>
<td>22.9</td>
<td>26.5</td>
<td>29.5</td>
<td>33.0</td>
<td>40.2</td>
<td>42.7</td>
<td>43.9</td>
<td>45.3</td>
<td>47.3</td>
<td>49.3</td>
<td>51.3</td>
<td>53.3</td>
<td>55.3</td>
<td>57.3</td>
<td>59.3</td>
<td>61.3</td>
<td>63.3</td>
<td>65.3</td>
<td>67.3</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

a. Write a polynomial function to model the data. Let \( L_1 \) be the number of years since 1900.

b. Estimate the median age of the population in 2050. \[ f(x) = 0.126x + 22.732 \]

Sample answer: \( f(x) = 0.126x + 22.732 \)

Sample answer: \( f(2050) \)

c. According to your model, in what year did the median age of the population reach 30? \[ f(x) = 0.126x + 22.732 \]

Sample answer: 1958

Tips for New Teachers

Correlation Coefficient Some students may need to turn on the correlation coefficient feature on their calculators. To do this from the home screen, press \( \text{2nd} \) \( \text{CATALOG} \), select \( \text{DiagnosticOn} \), and then press \( \text{ENTRY} \).

Additional Example

7. POPULATION The table below shows a town’s population over an 8-year period. Year 1 refers to the year 2001, year 2 refers to the year 2002, and so on.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5050</td>
</tr>
<tr>
<td>2</td>
<td>5510</td>
</tr>
<tr>
<td>3</td>
<td>5608</td>
</tr>
<tr>
<td>4</td>
<td>5496</td>
</tr>
<tr>
<td>5</td>
<td>5201</td>
</tr>
<tr>
<td>6</td>
<td>5089</td>
</tr>
<tr>
<td>7</td>
<td>5095</td>
</tr>
<tr>
<td>8</td>
<td>4675</td>
</tr>
</tbody>
</table>

a. Create a scatter plot of the data, and determine the type of polynomial function that could be used to represent the data. \[ f(x) = \text{cubic function} \]

b. Write a polynomial function to model the data set. Round each coefficient to the nearest thousandth, and state the correlation coefficient. \[ f(x) = 10.02x^3 - 176.32x^2 + 807.469x + 4454.786; \] \[ r^2 = 0.89. \]

Sample answer: \( f(x) = 10.02x^3 - 176.32x^2 + 807.469x + 4454.786; \]

Sample answer: 0.89

c. Use the model to estimate the population of the town in the year 2012. \[ f(x) = 10.02x^3 - 176.32x^2 + 807.469x + 4454.786; \]

Sample answer: 6069

d. Use the model to determine the approximate year in which the population reaches 10,712. \[ f(x) = 10.02x^3 - 176.32x^2 + 807.469x + 4454.786; \]

Sample answer: 2015
Graph each function.  
1. \( f(x) = (x + 5)^2 \)
2. \( f(x) = (x - 6)^3 \)
3. \( f(x) = x^4 - 6 \)
4. \( f(x) = x^3 + 7 \)
5. \( f(x) = (2x)^4 \)
6. \( f(x) = (2x)^3 - 16 \)
7. \( f(x) = (x - 3)^4 + 6 \)
8. \( f(x) = (x + 4)^3 - 3 \)
9. \( f(x) = \frac{1}{3}(x - 9)^3 \)
10. \( f(x) = \left(\frac{1}{2}x\right)^3 + 8 \)

11. Water. If it takes exactly one minute to drain a 10-gallon tank of water, the volume of water remaining in the tank can be approximated by \( v(t) = 10(1 - t)^2 \), where \( t \) is time in minutes, \( 0 \leq t \leq 1 \). Graph the function. (Example 1)

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. (Example 2)

12. \( f(x) = -5x^2 + 6x^4 + 8 \)
13. \( f(x) = 2x^6 + 4x^3 + 9x^2 \)
14. \( g(x) = 5x^5 + 7x^3 - 9 \)
15. \( g(x) = -7x^5 + 8x^4 - 6x^3 \)
16. \( h(x) = 8x^4 + 5 - 4x^3 \)
17. \( h(x) = 4x^2 + 5x^3 - 2x^5 \)
18. \( f(x) = x(x + 1)(x - 3) \)
19. \( g(x) = x^2(x + 4)(-2x + 1) \)
20. \( f(x) = -x(x - 4)(x + 5) \)
21. \( g(x) = x^3(x + 1)(x^2 - 4) \)

a–b. See Chapter 2 Answer Appendix.

22. Organic Food. The number of acres in the United States used for organic apple production from 2000 to 2005 can be modeled by \( a(x) = 43.77x^4 - 498.76x^3 + 1310.2x^2 + 1626.2x + 6821.5 \), where \( x = 0 \) in 2000. (Example 2)

a. Graph the function using a graphing calculator.

b. Describe the end behavior of the graph of the function using limits. Explain using the leading term test.

23–32. See Chapter 2 Answer Appendix.

State the number of possible real zeros and turning points of each function. Then determine all of the real zeros by factoring. (Examples 3–5)

23. \( f(x) = x^3 + 3x^4 + 2x^3 \)
24. \( f(x) = x^6 - 8x^5 + 12x^4 \)
25. \( f(x) = x^4 + 4x^2 - 21 \)
26. \( f(x) = x^4 - 4x^3 - 32x^2 \)
27. \( f(x) = x^5 - 6x^4 - 16 \)
28. \( f(x) = 4x^4 + 16x^4 + 12 \)
29. \( f(x) = 9x^4 - 36x^3 \)
30. \( f(x) = 6x^3 - 150x^3 \)
31. \( f(x) = 4x^4 - 4x^3 - 3x^2 \)
32. \( f(x) = 3x^3 + 11x^4 - 20x^3 \)

33–42. See Chapter 2 Answer Appendix.

For each function, (a) apply the leading-term test, (b) determine the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function. (Example 6)

33. \( f(x) = x(x + 4)(x - 1)^2 \)
34. \( f(x) = x^2(x - 4)(x + 2) \)
35. \( f(x) = -(x + 3)(x - 5)^3 \)
36. \( f(x) = 2x(x + 5)(x - 3) \)
37. \( f(x) = -(x - 3)(x + 2)^3 \)
38. \( f(x) = -(x^2)(x - 4)^2 \)
39. \( f(x) = 3x^3 - 3x^2 - 36x \)
40. \( f(x) = -2x^3 - 4x^2 + 6x \)
41. \( f(x) = x^4 + x^3 - 20x^2 \)
42. \( f(x) = x^5 + 3x^4 - 10x^3 \)

43. Reservoirs. The number of feet below the maximum water level in Wisconsin’s Rainbow Reservoir during ten months in 2007 is shown. (Example 7)

<table>
<thead>
<tr>
<th>Month</th>
<th>Level</th>
<th>Month</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>4</td>
<td>July</td>
<td>9</td>
</tr>
<tr>
<td>February</td>
<td>5.5</td>
<td>August</td>
<td>11</td>
</tr>
<tr>
<td>March</td>
<td>10</td>
<td>September</td>
<td>16.5</td>
</tr>
<tr>
<td>April</td>
<td>9</td>
<td>November</td>
<td>11.5</td>
</tr>
<tr>
<td>May</td>
<td>7.5</td>
<td>December</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Source: Wisconsin Valley Improvement Company

See margin.

a. Write a model that best relates the water level as a function of the number of months since January.

b. Use the model to estimate the water level in the reservoir in October. 14.8 ft

44–47. See Chapter 2 Answer Appendix.

Use a graphing calculator to write a polynomial function to model each set of data. (Example 7)

44.
\[ f(x) \]
\[ \begin{array}{c|cccc}
\hline
x & -3 & -2 & -1 & 0 \\
\hline
f(x) & 8.75 & 7.5 & 6.25 & 5 \\
\hline
\end{array} \]

45.
\[ f(x) \]
\[ \begin{array}{c|cccc}
\hline
x & 5 & 7 & 8 & 10 \\
\hline
f(x) & 2 & 5 & 6 & 4 \\
\hline
\end{array} \]

46.
\[ f(x) \]
\[ \begin{array}{c|cccc}
\hline
x & -5.5 & -2 & -1.5 & -1 \\
\hline
f(x) & 23 & 17 & 7 & 6 \\
\hline
\end{array} \]

47.
\[ f(x) \]
\[ \begin{array}{c|cccc}
\hline
x & 30 & 35 & 40 & 45 \\
\hline
f(x) & 52 & 41 & 32 & 34 \\
\hline
\end{array} \]

48a. Sample answer: \( f(x) = 2.14 \cdot 10^{-6}x^2 - 0.018x^2 + 0.383x + 5.976 \)

48b. Electricity. The average retail electricity prices in the U.S. from 1970 to 2005 are shown. Projected prices for 2010 and 2020 are also shown. (Example 7)

<table>
<thead>
<tr>
<th>Year</th>
<th>Price (¢/kWh)</th>
<th>Year</th>
<th>Price (¢/kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>7</td>
<td>2000</td>
<td>6.625</td>
</tr>
<tr>
<td>1980</td>
<td>7.25</td>
<td>2005</td>
<td>6.25</td>
</tr>
<tr>
<td>1982</td>
<td>9.625</td>
<td>2010</td>
<td>6.25</td>
</tr>
<tr>
<td>1990</td>
<td>8</td>
<td>2020</td>
<td>6.375</td>
</tr>
</tbody>
</table>

Source: Energy Information Administration

a. Write a model that relates the price as a function of the number of years since 1970.

b. Use the model to predict the average price of electricity in 2015. 5.93¢

c. According to the model, during which year was the price 7¢ for the second time? 1999
Determine whether each graph could show a polynomial function. Write yes or no. If not, explain why not.

50. Yes; there is a sharp turn at \( x = 2 \).
51. Yes; undefined at \( x = 0 \).
52. No; there is a sharp turn at \( x = 2 \).
53. No; undefined at \( x = 0 \).

54–63. See margin.
Find a polynomial function of degree \( n \) with only the following real zeros. More than one answer is possible.

54. \(-1; n = 3\)
55. \(3; n = 3\)
56. \(-3; n = 4\)
57. \(-5; 4; n = 4\)
58. \(7; n = 4\)
59. \(-4; n = 5\)
60. \(2, 1, 4; n = 5\)
61. \(0, 3, -2; n = 5\)
62. no real zeros; \( n = 4\)
63. no real zeros; \( n = 6\)

Determine whether the degree \( n \) of the polynomial for each graph is even or odd and whether its leading coefficient \( a_n \) is positive or negative.

64. \(n\) is even; \( a_n \) is positive.
65. \(n\) is odd; \( a_n \) is positive.
66. \(n\) is odd; \( a_n \) is negative.
67. \(n\) is even; \( a_n \) is negative.
68. MANUFACTURING A company manufactures aluminum containers for energy drinks.
   a. Write an equation \( V \) that represents the volume of the container.
   \[ V = \pi r^2 h \]
   b. Write a function \( A \) in terms of \( r \) that represents the surface area of a container with a volume of 15 cubic inches.
   c. Use a graphing calculator to determine the minimum possible surface area of the can.

69–74. See Chapter 2 Answer Appendix.
Determine a polynomial function that has each set of zeros. More than one answer is possible.
69. \(5, -3, 6\)
70. \(4, -8, -2\)
71. \(3, 0, 4, -1, 3\)
72. \(1, 1, -4, 6, 0\)
73. \(\frac{3}{4}, -3, -4, -\frac{2}{3}\)
74. \(-1, -1, 5, 0, \frac{2}{5}\)

75. POPULATION The percent of the United States population living in metropolitan areas has increased.

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent of Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>56.1</td>
</tr>
<tr>
<td>1960</td>
<td>63</td>
</tr>
<tr>
<td>1970</td>
<td>68.6</td>
</tr>
<tr>
<td>1980</td>
<td>74.8</td>
</tr>
<tr>
<td>1990</td>
<td>74.8</td>
</tr>
<tr>
<td>2000</td>
<td>79.2</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

a. Sample answer: \( f(x) = 0.45x + 88.19 \)
b. Sample answer: 2010

43a. Sample answer: \( f(x) = -0.018x^4 + 0.355x^3 - 2.276x^2 + 5.722x + 3.509 \)
54. Sample answer: \( f(x) = x^3 + 3x^2 + 3x + 1 \)
55. Sample answer: \( f(x) = x^3 - 9x^2 + 27x - 27 \)
56. Sample answer: \( f(x) = x^4 - 6x^3 - 27x^2 + 108x + 324 \)
57. Sample answer: \( f(x) = x^4 + 11x^3 + 15x^2 - 175x - 500 \)
58. Sample answer: \( f(x) = x^4 - 28x^3 + 294x^2 - 1372x + 2401 \)
59. Sample answer: \( f(x) = x^5 + 4x^4 \)
60. Sample answer: \( f(x) = x^5 - 13x^4 + 64x^3 - 148x^2 + 160x - 64 \)
61. Sample answer: \( f(x) = x^5 - x^4 - 6x^3 \)
62. Sample answer: \( f(x) = x^4 + 2x^2 + 1 \)
63. Sample answer: \( f(x) = x^6 + 3x^4 + 3x^2 + 1 \)

Watch Out!

Common Error For Exercise 49, you may wish to explain to students how the quarters relate to the years 2005–2007. Some students may mistakenly equate quarter 1 with 2005, quarter 2 with 2006, and so on. Each year has four quarters, so quarters 1–4 are for 2005, quarters 5–8 are for 2006, and 9–12 are for 2007.
Create a function with the following characteristics.

Graph the function.  76–79. See Chapter 2 Answer Appendix.

76. degree = 5, 3 real zeros, \( \lim_{x \to -\infty} = -\infty \) and \( \lim_{x \to \infty} = \infty \)
77. degree = 6, 4 real zeros, \( \lim_{x \to -\infty} = -\infty \) and \( \lim_{x \to \infty} = -\infty \)
78. degree = 5, 2 distinct real zeros, 1 of which has a multiplicity of 2, \( \lim_{x \to -\infty} = -\infty \) and \( \lim_{x \to \infty} = \infty \)
79. degree = 6, 3 distinct real zeros, 1 of which has a multiplicity of 2, \( \lim_{x \to -\infty} = -\infty \) and \( \lim_{x \to \infty} = \infty \)

a. Graph the data.
b. Use a graphing calculator to model the data using a polynomial function with a degree of 3.
c. Repeat part b using a function with a degree of 4.
d. Which function is a better model? Explain.

For each of the following graphs: 81–84. See margin.

a. Determine the degree and end behavior.
b. Locate the zeros and their multiplicity. Assume all of the zeros are integral values.
c. Use the given point to determine a function that fits the graph.

81. 82. 83. 84. 85–88. See margin.
State the number of possible real zeros and turning points of each function. Then find all of the real zeros by factoring.

85. \( f(x) = 16x^4 + 72x^2 + 80 \)
86. \( f(x) = -12x^3 - 44x^2 - 40x-8 \)
87. \( f(x) = -24x^4 + 24x^3 - 6x^2 - 4x - 24 \)
88. \( f(x) = x^3 + 6x^2 - 4x - 24 \)

90. MULTIPLE REPRESENTATIONS In this problem, you will investigate the behavior of combinations of polynomial functions.

a. GRAPHICAL Graph \( f(x) \), \( g(x) \), and \( h(x) \) in each row on the same graphing calculator screen. For each graph, modify the window to observe the behavior both on a large scale and very close to the origin. a-d. See Chapter 2 Answer Appendix.

b. ANALYTICAL Describe the behavior of each graph of \( f(x) \) in terms of \( g(x) \) or \( h(x) \) near the origin.
c. ANALYTICAL Describe the behavior of each graph of \( f(x) \) in terms of \( g(x) \) or \( h(x) \) as \( x \) approaches \( -\infty \) and \( \infty \).
d. VERBAL Predict the behavior of a function that is a combination of two functions \( a \) and \( b \) such that \( f(x) = a + b \), where \( a \) is the term of higher degree.

H.O.T. Problems Use Higher-Order Thinking Skills

90. ERROR ANALYSIS Colleen and Martin are modeling the data shown. Colleen thinks the model should be \( f(x) = 5.754x^3 + 2.912x^2 - 7.516x + 0.349 \). Martin thinks it should be \( f(x) = 3.697x^2 + 11.734x - 2.476 \). Is either of them correct? Explain your reasoning.

91. REASONING Can a polynomial function have both an absolute maximum and an absolute minimum? Explain your reasoning. See Chapter 2 Answer Appendix.

92. REASONING Explain why the constant function \( f(x) = c \), \( c \neq 0 \), has degree 0, but the zero function \( f(x) = 0 \) has no degree. See Chapter 2 Answer Appendix.

93. CHALLENGE Use factoring by grouping to determine the zeros of \( f(x) = x^3 + 7x^2 - 5x - 12x - 60 \). Explain each step. See Chapter 2 Answer Appendix.

94. REASONING How is it possible for more than one function to be represented by the same degree, end behavior, and distinct real zeros? Provide an example to explain your reasoning. See Chapter 2 Answer Appendix.

95. REASONING What is the minimum degree of a polynomial function that has an absolute maximum, a relative maximum, and a relative minimum? Explain your reasoning.

96. WRITING IN MATH Explain how you determine the best polynomial function to use when modeling data.

**Additional Answers**

81a. degree = 4; \( \lim_{x \to \infty} = \infty \) and \( \lim_{x \to -\infty} = \infty \)
81b. \(-6, -2 \) (multiplicity: 2), 4
81c. Sample answer: \( f(x) = 0.5(x + 2)^2 \) \((x + 6)(x - 4)\)
82a. degree = 4; \( \lim_{x \to \infty} = -\infty \) and \( \lim_{x \to -\infty} = -\infty \)
82b. \(-3, 2, 6 \) (multiplicity: 2)
Spiral Review

Solve each equation. (Lesson 2-1)
97. \( \sqrt{x + 3} = 7 \) 66
98. \( d + \sqrt{d - 8} = 4 \) 3
99. \( \sqrt{a - 8} = \sqrt{13 + a} \) no solution

100. REMODELING An installer is replacing the carpet in a 12-foot by 15-foot living room.
The new carpet costs \$13.99 per square yard. The formula \( f_1(x) = 9x \) converts square yards to
square feet. (Lesson 1-7)
a. Find the inverse \( f^{-1}(x) \). What is the significance of \( f^{-1}(x) \)?
b. How much will the new carpet cost? \$279.80
Given \( f_2(x) = 2x^2 - 5x + 3 \) and \( g(x) = 6x + 4 \), find each function.
101. \( f + g(x) \) 2\( x^2 + x + 7 \)
102. \( f \cdot g(x) \) 72\( x^2 + 66x + 15 \)
103. \( g \cdot f(x) \) 12\( x^2 - 30x + 22 \)

Describe how the graphs of \( f(x) = x^2 \) and \( g(x) \) are related. Then write an equation for \( g(x) \). (Lesson 1-5) 104-106. See Chapter 2 Answer Appendix.

107. BUSINESS A company creates a new product that costs \$25 per item to produce. They hire a
marketing analyst to help determine a selling price. After collecting and analyzing data
relating selling price \( s \) to yearly consumer demand \( d \), the analyst estimates demand for the
product using \( d = -200s + 15,000 \). (Lesson 1-6)
a. If yearly profit is the difference between total revenue and production costs,
determine a selling price \( s \geq 25 \), that will maximize the company’s yearly profit \( P \).
(Hint: \( P = sd - 25d \)) \$50
b. What are the risks of determining a selling price using this method?

Sample answer: The company’s competition
might offer a similar product at a lower cost.

Skills Review for Standardized Tests

108. SAT/ACT The figure shows the
intersection of three lines. The
figure is not drawn to scale.
\( x = \)
A 16  D 60
B 20  E 90
C 30

109. Over the domain \( 2 < x \leq 3 \), which of the following functions contains the greatest values of \( y \)?
F \( y = \frac{x + 3}{x - 2} \)
G \( y = \frac{x - 5}{x + 1} \)
H \( y = x^2 - 3 \)
J \( y = 2x \)

110. MULTIPLE CHOICE Which of the following equations represents the result of shifting the parent function
\( y = x^3 \) up 4 units and right 5 units?  D
A \( y + 4 = (x + 5)^3 \)
B \( y - 4 = (x + 5)^3 \)
C \( y + 4 = (x - 5)^3 \)
D \( y - 4 = (x - 5)^3 \)

111. REVIEW Which of the following describes the numbers
in the domain of \( h(x) = \sqrt{2x - 3} \) \( x - 5 \)?
F \( x \neq 5 \)
G \( x \geq \frac{3}{2} \)
H \( x \geq \frac{3}{2} \), \( x \neq 5 \)
J \( x \neq \frac{3}{2} \)

Name the Math Ask students to
describe the graph of a polynomial function. Sample answer: a smooth
curve that is defined and continuous for all real numbers

Additional Answers
82c. Sample answer: \( f(x) = -0.25 \)
\( (x - 6)^2 (x - 2)(x + 3) \)
83a. degree = 4; \( \lim_{x \to -\infty} = -\infty \) and \( \lim_{x \to \infty} = \infty \)
83b. \(-4, 3 \) (multiplicity: 2), \(-1 \)
83c. Sample answer: \( f(x) = \frac{1}{8} (x - 3)^2 \)
\( (x + 1)(x + 4) \)
84a. degree = 5; \( \lim_{x \to -\infty} = -\infty \) and \( \lim_{x \to \infty} = \infty \)
84b. \(-3 \) (multiplicity: 2), \(-1, 2 \)
(multiplicity: 2)
84c. Sample answer: \( f(x) = \frac{1}{10} \)
\( (x - 2)^2 (x + 1)(x + 3)^2 \)
85. 4 real zeros and 3 turning points; no real zeros
86. 3 real zeros and 2 turning points; 0, \(-2, \) and \(\frac{5}{3} \)
87. 4 real zeros and 3 turning points; 0, \(-1\), and \(\frac{1}{2} \)
88. 3 real zeros and 2 turning points;
\(-6, -2, \) and \(2 \)

Extension Pose the following question to students. Is it always appropriate to use a polynomial model
for a scatter plot in order to predict beyond the range of the data given? Ask students to give examples
to support their responses. No, because realistic conditions may not exist beyond the data given;
Sample answer: a scatter plot of sports records may show that the record time for completing a
100-meter race has decreased over the years. The model for the scatter plot will decrease indefinitely,
becoming zero, then negative. However, the race will never be run in zero or negative time.
1 Focus

Objective  Use TI-Nspire technology to explore the hidden behavior of graphs.

Teaching Tip
The graphing calculator opens on the same screen as when it was turned off. Have students press c to begin the lab.

2 Teach

Working in Cooperative Groups  Pair students with different abilities. Have students work through Activity Steps 1–3.

Ask:
- How many possible real zeros can a cubic function have?  1, 2, or 3
- If this function has only one real zero, what type of zeros are the other two? imaginary zeros
- How many possible real zeros can a quartic function have?  0, 1, 2, 3, or 4

Practice  Have students work through Exercises 3–6.

3 Assess

Formative Assessment  Use Exercise 6 to assess whether students can use TI-Nspire technology to graph functions and find zeros.

From Concrete to Abstract  Ask students to describe when they are certain that a value is a zero of a graphed function.

Extending the Concept  Ask students how they could keep the graph about the same in Step 1 but have either two or three zeros in Step 3.

Activity 1  Hidden Behavior of Graphs
Determine the zeros of \( f(x) = x^3 - x^2 - 60.7x + 204 \) graphically.

Step 1  Open a new Graphs and Geometry page, and graph the function.

In the default window, it appears that the function has two zeros, one between \(-10\) and \(-8\) and one between 4 and 6.

Step 2  From the Window menu, choose Window Settings. Change the dimensions of the window as shown.

The behavior of the graph is much clearer in the larger window. It still appears that the function has two zeros, one between \(-8\) and \(-10\) and one between 4 and 6.

Step 3  From the Window menu, choose Window Settings. Change the window to \([2, 8] \times [-2, 2]\).

By enlarging the graph in the area where it appears that the zero occurs, it is clear that there is no zero between the values of 4 and 6. Therefore, the graph only has one zero.

Analyze the Results

1. In addition to the limitation discovered in the previous steps, how can graphing calculators limit your ability to interpret graphs?
2. What are some ways to avoid these limitations?

Exercises
Determine the zeros of each polynomial graphically. Watch for hidden behavior.

3. \( x^2 + 6.5x^2 - 46.5x + 60 = -11.2, 2.2, 2.5 \)
4. \( x^4 - 3x^3 + 12x^2 + 6x - 7 = -0.89, 0.58 \)
5. \( x^5 + 7x^3 + 4x^2 = x + 10.9 = -1.3 \)
6. \( x^4 - 19x^3 + 107.2x^2 - 162x + 73 = 8.1, 8.9 \)
Divide Polynomials Consider the polynomial function \( f(x) = 6x^3 - 25x^2 + 18x + 9 \). If you know that \( f \) has a zero at \( x = 3 \), then you also know that \( (x - 3) \) is a factor of \( f(x) \). Because \( f(x) \) is a third-degree polynomial, you know that there exists a second-degree polynomial \( q(x) \) such that

\[
f(x) = (x - 3) \cdot q(x).
\]

This implies that \( q(x) \) can be found by dividing \( 6x^3 - 25x^2 + 18x + 9 \) by \( (x - 3) \) because

\[
q(x) = \frac{f(x)}{x - 3}, \text{ if } x \neq 3.
\]

To divide polynomials, we can use an algorithm similar to that of long division with integers.

**Example 1** Use Long Division to Factor Polynomials

Factor \( 6x^3 - 25x^2 + 18x + 9 \) completely using long division if \( (x - 3) \) is a factor.

\[
\begin{align*}
6x^3 - 7x - 3 &= \frac{6x^3 - 25x^2 + 18x + 9}{x - 3} \quad \text{Multiply divisor by } 6x^2 \text{ because } \frac{6x^3}{x} = 6x^2. \\
&= \frac{6x^3}{x} - \frac{25x^2}{x} + \frac{18x}{x} + \frac{9}{x} \\
&= 6x^2 - 25x + 18x + 9 \\
&= -7x^2 + 18x \\
&= \frac{-7x^2}{x} + \frac{18x}{x} \\
&= -7x + 18x \\
&= 11x \\
&= \frac{11x}{x} \\
&= 11 \\
&= \frac{-3x + 9}{x} \\
&= -\frac{3x}{x} + \frac{9}{x} \\
&= -3 + 9 \\
&= 6 \\
&= \frac{6}{x} \\
&= 0
\end{align*}
\]

From this division, you can write \( 6x^3 - 25x^2 + 18x + 9 = (x - 3)(6x^2 - 7x - 3) \).

Factoring the quadratic expression yields \( 6x^3 - 25x^2 + 18x + 9 = (x - 3)(2x - 3)(3x + 1) \).

So, the zeros of the polynomial function \( f(x) = 6x^3 - 25x^2 + 18x + 9 \) are 3, \( \frac{3}{2} \), and \( -\frac{1}{3} \). The \( x \)-intercepts of the graph of \( f(x) \) shown support this conclusion.

**Guided Practice**

Factor each polynomial completely using the given factor and long division.

1A. \( x^2 + 7x^2 + 4x - 12; x + 6 \)
1B. \( 6x^3 - 2x^2 - 16x - 8; 2x - 4 \) \( (x + 1)(2x - 4)(3x + 2) \)
Practice: Suppose the polynomial function that represents the average height of redwood trees as they age is \( f(x) = -0.001x^2 + 1.15x + 8 \). How could you estimate the height of a redwood tree that is 1000 years old? Substitute 1000 for \( x \) and evaluate.

Practice: How many possible real zeros does this function have? Explain how you know. 2; The degree of the polynomial is 2.

1 Divide Polynomials

Examples 1–4 show how to divide polynomials using long division and synthetic division. This includes the need for the polynomials to be in standard form and to use zero coefficients for missing powers as place holders.

Formative Assessment
Use the Guided Practice exercises after each example to determine students’ understanding of concepts.

Additional Examples

1. Factor \( 6x^3 + 17x^2 - 104x + 60 \) completely using long division if \( (2x - 5) \) is a factor.

2. Divide \( 6x^3 - 5x^2 + 9x + 6 \) by \( 2x - 1 \).

Additional Examples also in Interactive Classroom PowerPoint® Presentations

StudyTip

Proper vs. Improper A rational expression is considered improper if the degree of the numerator is greater than or equal to the degree of the denominator. So in the division algorithm, \( \frac{p(x)}{q(x)} \) is an improper rational expression, while \( \frac{q(x)}{p(x)} \) is a proper rational expression.

Long division of polynomials can result in a zero remainder, as in Example 1, or a nonzero remainder, as in the example below. Notice that just as with integer long division, the result of polynomial division is expressed using the quotient, remainder, and divisor.

Recall that a dividend can be expressed in terms of the divisor, quotient, and remainder.

This leads to a definition for polynomial division.

Before dividing, be sure that each polynomial is written in standard form and that placeholders with zero coefficients are inserted where needed for missing powers of the variable.

Example 2 Long Division with Nonzero Remainder

Divide \( 9x^2 - x - 3 \) by \( 3x + 2 \).

First rewrite \( 9x^2 - x - 3 \) as \( 9x^2 + 0x^1 - x - 3 \). Then divide.

You can write this result as \( \frac{3x^2 - 2x + 1}{3x + 2} = \frac{3x^2 - 2x + 1 + \frac{5}{3x + 2}}{3x + 2} \neq \frac{3x^2 - 2x + 1 - \frac{5}{3x + 2}}{3x + 2} \).

CHECK Multiply to check this result.

\[ (3x + 2)(3x^2 - 2x + 1) + (-5) = 9x^3 - x - 3 \]
\[ 9x^3 - 6x^2 + 3x + 6x^2 - 4x + 2 - 5 = 9x^3 - x - 3 \]
\[ 9x^3 - 9x^3 - x - 3 \checkmark \]

GuidedPractice

Divide using long division.

2A. \( 4x^2 - 3x + 6 - \frac{2}{x} \neq \frac{3}{2} \)

2B. \( (8x^3 - 18x^2 + 21x - 20) ÷ (2x - 3) \)

When dividing polynomials, the divisor can have a degree higher than 1. This can sometimes result in a quotient with missing terms.
Division by Zero
In Example 3, this division is not defined for 

toward in this lesson, you can 

value for which the indicated 

division is undefined.

Guided Practice
Divide using long division.

Example 3
Division by Polynomial of Degree 2 or Higher

Divide $2x^3 - 4x^2 + 13x^2 + 3x - 11$ by $y = 2x^2 - 7x - 9$.

\[
\begin{align*}
2x^2 & - 7x - 3 \\
-3x^3 & - 18x^2 \\
-7x^2 & + 18x \\
-3x + 9 \\
\hline
0 & 
\end{align*}
\]

You can write this result as 

\[
\frac{2x^2 - 4x^3 + 13x^2 + 3x - 11}{y^2 - 2x + 7} = 2x^2 - 1 + \frac{-x - 4}{y^2 - 2x + 7}
\]

Guided Practice
Divide using long division.

3A. $(2x^3 + 5x^2 - 7x + 9) + (x^2 + 3x - 4)$

3B. $(6x^3 - x^4 + 12x^2 + 15x) + (3x^3 - 2x^2 + x)$

Synthetic division is a shortcut for dividing a polynomial by a linear factor of the form $x - c$.

Consider the long division from Example 1.

<table>
<thead>
<tr>
<th>Long Division</th>
<th>Supress Variables</th>
<th>Collapse Vertically</th>
<th>Synthetic Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notice the coefficients highlighted in colored text.</td>
<td>Suppress x and powers of x.</td>
<td>Collapse the long division vertically, eliminating duplications.</td>
<td>Change the signs of the divisor and the numbers on the second line.</td>
</tr>
<tr>
<td>$6x^2 - 7x - 3$</td>
<td>$- 3x^3 - 18x^2$</td>
<td>$-3x^2 + 21x$</td>
<td>$3$</td>
</tr>
<tr>
<td>$-7x^2 + 18x$</td>
<td>$-3x + 9$</td>
<td>$-3 + 9$</td>
<td>$6 - 25 18 9$</td>
</tr>
<tr>
<td>$(-) -3x + 9$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

We can use the synthetic division shown in the example above to outline a procedure for synthetic division of any polynomial by a binomial.

Key Concept
Synthetic Division Algorithm

To divide a polynomial by the factor $x - c$, complete each step.

Step 1 Write the coefficients of the dividend in standard form. Write the related zero $c$ of the divisor $x - c$ in the box. Bring down the first coefficient.

Step 2 Multiply the first coefficient by $c$. Write the product under the second coefficient.

Step 3 Add the product and the second coefficient.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Example Divide $6x^3 - 25x^2 + 18x + 9$ by $x - 3$.

\[
\begin{array}{cccc}
6 & -25 & 18 & 9 \\
\hline
18 & -3 & 0 & \\
\end{array}
\]

\[
\begin{array}{cccc}
6 & -25 & 18 & 9 \\
18 & -3 & 0 & \\
\end{array}
\]

Focus on Mathematical Content

Synthetic Division Synthetic Division is a method of dividing a polynomial by a factor in the form $(x - c)$. For synthetic division to work, the polynomial must be written in standard form with zeros as placeholders for any missing powers. The divisor must be in the form $(x - c)$. For polynomials being divided by divisors that have a degree 2 or higher, long division is used.
Teach with Tech

Wiki Have students create a wiki page explaining how to set up the synthetic division of a division problem involving polynomials. Be sure that they explain how they suppress the variables as well as how they change the signs of the divisor and the numbers on the second line.

2 The Remainder and Factor Theorems

Example 5 shows how to use the Remainder Theorem. Example 6 shows how to use the Factor Theorem to determine if \( x - c \) is a factor of a polynomial \( f(x) \).

As with division of polynomials by long division, remember to use zeros as placeholders for any missing terms in the dividend. When a polynomial is divided by one of its binomial factors \( x - c \), the quotient is called a depressed polynomial.

Example 4 Synthetic Division

Divide using synthetic division.

**Example 4 Synthetic Division**

Divide using synthetic division.

- (2x^4 - 5x^3 + 5x - 2) ÷ (x + 2)
  - Because \( x + 2 = x - (-2) \), \( c = -2 \). Set up the synthetic division as follows, using zero as a placeholder for the missing \( x^3 \)-term in the dividend. Then follow the synthetic division procedure.

```
     2      0    -5    5    -2
-2 | 2  0   -10  -20  42

     2      0    -5    5    -2
```

The quotient has degree one less than that of the dividend, so

\[
\frac{2x^4 - 5x^3 + 5x - 2}{x + 2} = 2x^3 - 4x^2 + 3x - 1.
\]

**Example 5**

**Example 5**

Rewrite the division expression so that the divisor is of the form \( x - c \).

\[
\frac{10x^3 - 13x^2 + 5x - 14}{2x - 3}
\]

So, \( c = \frac{3}{2} \). Perform the synthetic division.

```
     5      -13    5    -7
\frac{3}{2} | 2 5 -13 5

     5      -13    5    -7
```

So

\[
\frac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = 5x^2 + x + \frac{1}{2}.
\]

Guided Practice

**Guided Practice**

**Guided Practice**

4A. \( (4x^3 + 3x^2 - x + 8) ÷ (x - 3) \)

2x^2 + 3x^2 - 6x - 2 + \frac{9}{3x + 1}

**Guided Practice**

2 The Remainder and Factor Theorems

When \( d(x) \) is the divisor \( (x - c) \) with degree 1, the remainder is the real number \( r \). So, the division algorithm simplifies to

\[
f(x) = (x - c) \cdot q(x) + r.
\]

Evaluating \( f(x) \) for \( x = c \), we find that

\[
f(c) = (c - c) \cdot q(c) + r = 0 \cdot q(c) + r \text{ or } r.
\]

So, \( f(c) = r \), which is the remainder. This leads us to the following theorem.

**Key Concept: Remainder Theorem**

If a polynomial \( f(x) \) is divided by \( x - c \), the remainder is \( r = f(c) \).

Tips for New Teachers

**Zero Coefficients** In Examples 2 and 4, emphasize the importance of writing each polynomial in standard form. Have students leave space in a long division problem or insert a zero in a synthetic division problem if any powers of \( x \) in the dividend have coefficients of zero. Have each student do a few synthetic division problems independently when the concept is first presented so that the necessary steps will be learned.
The Remainder Theorem indicates that to evaluate a polynomial function \( f(x) \) for \( x = c \), you can divide \( f(x) \) by \( x - c \) using synthetic division. The remainder will be \( f(c) \). Using synthetic division to evaluate a function is called **synthetic substitution**.

### Use the Remainder Theorem

**FOOTBALL** The number of tickets sold during the Northside High School football season can be modeled by \( f(t) = t^3 - 12t^2 + 48t + 74 \), where \( t \) is the number of games played. Use the Remainder Theorem to find the number of tickets sold during the twelfth game of the Northside High School football season.

To find the number of tickets sold during the twelfth game, use synthetic substitution to evaluate \( f(t) \) for \( t = 12 \).

```
  12 1 -12 48 74
  1 12 0 576
---
  1 0 48 650
```

**CHECK** You can check your answer using direct substitution.

\[
f(t) = t^3 - 12t^2 + 48t + 74
\]

\[
f(12) = (12)^3 - 12(12)^2 + 48(12) + 74
\]

**Guided Practice**

5. **FOOTBALL** Use the Remainder Theorem to determine the number of tickets sold during the thirteenth game of the season. 867

If you use the Remainder Theorem to evaluate \( f(x) \) at \( x = c \) and the result is \( f(c) = 0 \), then you know that \( c \) is a zero of the function and \( (x - c) \) is a factor. This leads us to another useful theorem that provides a test to determine whether \( (x - c) \) is a factor of \( f(x) \).

### Factor Theorem

A polynomial \( f(x) \) has a factor \( (x - c) \) if and only if \( f(c) = 0 \).

You can use synthetic division to perform this test.

### Use the Factor Theorem

Use the Factor Theorem to determine if the binomials given are factors of \( f(x) \). Use the binomials that are factors to write a factored form of \( f(x) \).

a. \( f(x) = x^3 - 18x^2 + 60x + 25; (x - 5), (x + 5) \)

\[ f(x) = (x - 5)(x^2 - 13x - 5) \]

b. \( f(x) = x^3 - 2x^2 - 13x - 10; (x - 5), (x + 2) \)

\[ f(x) = (x - 5)(x + 2)(x + 1) \]

### Tips for New Teachers

**Remainder Theorem** Explain to students that synthetic substitution is performed exactly like synthetic division. However, the last line of numbers is interpreted differently. Synthetic division uses the entire last line of numbers for the quotient and remainder. For synthetic substitution, only the last number in the last line is of interest, since that is the number that gives the value of the function for \( x = c \).
The Remainder and Factor Theorems

13. \(2x^3 - x^2 - 41x = 20; (x + 4), (x - 5)\)

Use synthetic division to test the factor \((x + 4)\).

\[
\begin{array}{c|cccc}
-4 & 2 & -1 & -41 & 20 \\
0 & 0 & 8 & 36 & 0 \\
2 & 2 & -9 & 0 & 0 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x + 4)\) is 0, \(f(-4) = 0\) and \((x + 4)\) is a factor of \(f(x)\).

Next, test the second factor, \((x - 5)\), with the depressed polynomial \(2x^2 - 9x - 5\).

\[
\begin{array}{c|cc}
5 & 2 & -9 \\
0 & 10 & 5 \\
2 & 1 & 0 & 0 \\
\end{array}
\]

Because the remainder when the quotient of \(f(x) ÷ (x - 5)\) is divided by \((x - 5)\) is 0, \(f(5) = 0\) and \((x - 5)\) is a factor of \(f(x)\).

Because \((x + 4)\) and \((x - 5)\) are factors of \(f(x)\), we can use the final quotient to write a factored form of \(f(x)\).

\[f(x) = (x + 4)(x - 5)(2x + 1)\]

CHECK The graph of \(f(x) = 2x^3 - x^2 - 41x = 20\) confirms that \(x = -4, x = 5, \) and \(x = -\frac{1}{2}\) are zeros of the function.

Guided Practice

Use the Factor Theorem to determine if the binomials given are factors of \(f(x)\). Use the binomials that are factors to write a factored form of \(f(x)\).

6A. \(f(x) = 3x^3 - x^2 - 22x + 24; (x - 2), (x + 5)\) yes; \(f(x) = (x - 2)(3x - 4)(x + 3)\)

6B. \(f(x) = 4x^3 - 34x^2 + 54x + 36; (x - 6), (x + 3)\) yes; \(f(x) = 2(x - 6)(x - 3)(2x + 1)\)

You can see that synthetic division is a useful tool for factoring and finding the zeros of polynomial functions.

Technology Tip

Zeros You can confirm the zeros on the graph of a function by using the zero feature on the CALC menu of a graphing calculator.
Factor each polynomial completely using the given factor and long division. (Example 1)

1. \(x^3 + 2x^2 - 23x - 60; x + 4\) \((x + 3)(x - 5)(x + 4)\)
2. \(x^3 + 2x^2 - 21x + 18; x - 3\) \((x - 1)(x + 6)(x - 3)\)
3. \(x^3 + 3x^2 - 18x - 40; x - 4\) \((x + 5)(x + 2)(x - 4)\)
4. \(4x^3 + 20x^2 - 8x + 16; x + 3\) \(4(x + 4)(x - 2)(x + 3)\)
5. \(-3x^3 + 15x^2 + 108x - 540; x - 6\) \((-3)(x - 5)(x + 6)(x - 6)\)
6. \(6x^3 - 7x^2 - 29x - 12; 3x + 4\) \((2x + 1)(3)(x + 3)(x + 4)\)
7. \(x^3 + 12x + 38x^2 + 126x - 26; 3x + 2 + 6x + 5\)
8. \(x^3 - 3x^2 - 6x^2 + 8x + 240; 2x^2 - 12\) \((x + 4)(x + 3)(x - 2)(x + 6)(x - 6)\)

Divide using long division. (Examples 2 and 3)
9. \((5x^3 + 3x + 6x^2 - x + 12) ÷ (x - 4)\) See margin.
10. \((x^6 - 2x^3 + x^2 - 3x^2 - 2x + 2) ÷ (x + 2)\)
11. \((4x^4 - 8x^3 + 12x^2 - 6x + 12) ÷ (x + 2)\)
12. \((2x^4 - 7x^3 - 38x + 103 + 60) ÷ (x - 3)\)
13. \((6x^4 - 3x^3 + 5x^2 - 2x + 10x - 6) ÷ (x + 2)\)
14. \((10x^5 - 3x^6 + 75x^2 + 36x + 24) ÷ (3x + 2)\)
15. \((x^4 + x^3 + 6x^2 + 18x - 216) ÷ (x^3 - 3x^2 + 18x - 54)\)
16. \((4x^4 - 14x^3 + 110x - 84) ÷ (2x^2 + x - 12)\)
17. \(6x^3 - 12x^2 + 21x^2 - 2x^2 + 8x + 8\)
18. \(3x^3 + 2x^2 + 3\)
19. \(12x^3 + 15x^2 + 19x^2 - 4x - 28\)
20. \(3x^2 + 2x^2 - 6x + 6\)

Divide using synthetic division. (Example 4)
21. \((x^3 - x^3 + 3x^2 - 6x - x + 12) ÷ (x - 2)\) See margin.
22. \((2x^4 + 4x^3 - 2x^2 + 8x - 4) ÷ (x + 3)\)
23. \((3x - 9x^2 - 24x - 48) ÷ (x - 4)\)
24. \((x^3 - 3x^2 + 6x^2 + 9x + 6) ÷ (x + 2)\)
25. \((12x - 3x^2 + 18x - 12x - 8) ÷ (2x - 3)\)
26. \((36x^2 + 3x^3 - 12x^2 - 30x - 12) ÷ (3x + 1)\)
27. \((45x^2 + 6x^2 + 3x^3 + 8x + 12) ÷ (3x - 2)\)
28. \((48x^3 + 28x^4 + 68x^3 + 11x + 6) ÷ (4x + 1)\)
29. \((60x^4 + 98x^3 + 9x^4 + 12x^2 = 25x - 20) ÷ (5x + 4)\)
30. \((16x^2 - 3x^4 - 24x^4 + 96x^4 = 42x^2 - 30x + 105) ÷ (x^2 - 7)\)

EDUCATION The number of U.S. students, in thousands, that graduated with a bachelor’s degree from 1970 to 2006 can be modeled by \(g(x) = 0.00002x^2 - 0.016x + 0.512x^3 - 7.15x + 47.52x + 800.27\), where \(x\) is the number of years since 1970. Use synthetic substitution to find the number of students that graduated in 2005. Round to the nearest thousand. (Example 5) 215,000 students

WatchOut! Synthetic Division For Exercises 19–28, watch for students who might use the wrong sign for the number in the synthetic division box. Remind students that the divisor must be written in the form \(x - c\). For example, a divisor such as \(x + 3\) must be written as \((x - (-3))\) to see that the number to use in the synthetic division box is \(-3\) and not \(3\).

### Additional Answers

23. \(6x^4 + 14x^3 + 12x^2 + 12x + 18 + \frac{46}{2x - 3}\)
24. \(12x^3 - 6x^2 + 6x - 12\)
25. \(15x^4 + 12x^3 + 9x^2 + 6x + \frac{20}{3} + \frac{76}{3(3x - 2)}\)
26. \(12x^4 + 4x^3 + 16x^2 - 4x + \frac{15}{4} + \frac{9}{4(4x + 1)}\)
27. \(12x^5 + 6x^4 - 3x^3 - 5\)
28. \(8x^5 - 12x^3 + 6x^2 - 15\)
29. no, no
30. yes, yes; \(f(x) = (x + 2)(x^3 - 4x^2 - x + 3)\)
31. yes, yes; \(f(x) = (x - 3)(x - 5)(x - 4)(x + 2)\)
32. yes, no; \(f(x) = (x - 4)(x^3 - 9x + 2)\)
33. yes, no; \(f(x) = (x - 3)(x^3 - 4x^2 - x + 3)\)
34. yes, no; \(f(x) = (x + 3)(x^3 + 2x^2 - 3x + 21)\)
Factor each polynomial using the given factor and long division. Assume $n > 0$.

48. $x^3 + 2x^2 - 14x - 24$; $x + 2$ (Equation 2.9) (Equation 2.3) (Equation 2.5)
49. $x^3 - x^2 - 12x + 10$; $x - 1$ (Equation 2.9) (Equation 2.3) (Equation 2.5)
50. $4x^3 + 2x^2 - 10x + 4$; $2x + 1$ (Equation 2.9) (Equation 2.3) (Equation 2.5)
51. $9x^3 + 24x^2 - 17x + 54$; $3x - 1$ (Equation 2.9) (Equation 2.3) (Equation 2.5)
52. MANUFACTURING

An 18-inch by 20-inch sheet of cardboard is cut and folded into a bakery box.

Find the value of $k$ so that each remainder is zero.

53. $x^3 - 2x^2 + 4x - 2$  54. $x^3 + 10x^2 + 2x + 4$  56. $x^3 - 2x^2 + 4x - 2$
55. $x^3 + 2x^2 + 4x + 2$  56. $2x^2 - x^2 + x + k$  57. SCULPTING

Esteban will use a block of clay that is 3 feet by 4 feet by 5 feet to make a sculpture. He wants to reduce the volume of the clay by removing the same amount from the length, the width, and the height.

a. Write a polynomial function to model the situation.

b. Graph the function. See margin.

c. The company wants to have a volume of 196 cubic inches. Write an equation to model this situation.

196 = $3x^2 - 47x^2 + 180x$

d. Find a positive integer for $x$ that satisfies the equation found in part c. 2 in.

b. NUMERICAL

Use synthetic division to evaluate each function in part a for three integer values greater than the greatest zero.

c. VERBAL

Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for an integer greater than its greatest zero.

d. NUMERICAL

Use synthetic division to evaluate each function in part a for three integer values less than the least zero.

e. VERBAL

Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for a number less than its least zero.

Sample answer: The elements in the last row alternate between nonnegative and nonpositive.
Spiral Review

Determine whether the degree $n$ of the polynomial for each graph is even or odd and whether its leading coefficient $a_n$ is positive or negative. (Lesson 2–2)

69. odd; negative
70. even; negative
71. odd; positive

72. SKYDIVING The approximate time $t$ in seconds that it takes an object to fall a distance of $d$ feet is given by $t = \sqrt{\frac{d}{16}}$. Suppose a skydiver falls 11 seconds before the parachute opens. How far does the skydiver fall during this time period? (Lesson 2–1) **1936 ft**

73. FIRE FIGHTING The velocity $v$ and maximum height $h$ of water being pumped into the air are related by $v = \sqrt{2gh}$, where $g$ is the acceleration due to gravity (32 feet/second$^2$). (Lesson 1–7)
   a. Determine an equation that will give the maximum height of the water as a function of its velocity.
   b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department’s needs? Explain.

   **Yes; the pump can propel water to a height of about 88 ft.**

Skills Review for Standardized Tests

74. SAT/ACT In the figure, an equilateral triangle is drawn with an altitude that is also the diameter of the circle. If the perimeter of the triangle is 36, what is the circumference of the circle? **B**
   A $6\sqrt{2}\pi$  C $12\sqrt{2}\pi$  E $36\pi$
   B $6\sqrt{3}\pi$  D $12\sqrt{3}\pi$

75. REVIEW If (3, −7) is the center of a circle and (8, 5) is on the circle, what is the circumference of the circle? **K**
   F $13\pi$  H $18\pi$  K $26\pi$
   G $15\pi$  J $25\pi$

76. REVIEW The first term in a sequence is $x$. Each subsequent term is three less than twice the preceding term. What is the 5th term in the sequence? **H**
   A $8x - 21$  C $16x - 39$  E $32x - 43$
   B $8x - 15$  D $16x - 45$

77. Use the graph of the polynomial function. Which is not a factor of $x^5 + x^4 - 3x^3 - 4x^2 - 4x - 4$? **H**
   F $(x - 2)$  G $(x + 2)$  H $(x - 1)$  J $(x + 1)$

票出教室

票学生要写剩下的 $x^3 - x^2 - 5x - 3$ 除以 $x - 3$.

**0**

**Ticket Out the Door** Ask students to write the remainder of $x^3 - x^2 - 5x - 3$ when it is divided by $x - 3$.

**Additional Answers**

57b. **Volume of Clay**

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Greatest Zero</th>
<th>Least Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3 - 2x^2 - 11x + 12$</td>
<td>4</td>
<td>−3</td>
</tr>
<tr>
<td>$x^4 + 6x^3 + 3x^2 - 10x$</td>
<td>1</td>
<td>−5</td>
</tr>
<tr>
<td>$x^5 - x^4 - 2x^3$</td>
<td>2</td>
<td>−1</td>
</tr>
</tbody>
</table>

56b. Sample answer: $f(x) = x^3 - 2x^2 - 11x + 12$: $f(5) = 32$, $f(7) = 180$, $f(8) = 308$; $f(x) = x^4 + 6x^3 + 3x^2 - 10x$: $f(2) = 56$, $f(3) = 240$, $f(4) = 648$; $f(x) = x^5 - x^4 - 2x^3$: $f(3) = 108$, $f(5) = 2250$, $f(7) = 13,720$

60b. Sample answer: All of the elements in the last row of the synthetic division are positive.

60c. Sample answer: $f(x) = x^3 - 2x^2 - 11x + 12$: $f(-4) = -40$, $f(-5) = -108$, $f(-6) = -210$; $f(x) = x^4 + 6x^3 + 3x^2 - 10x$: $f(-6) = 168$, $f(-7) = 560$, $f(-9) = 2520$; $f(x) = x^5 - x^4 - 2x^3$: $f(-2) = -32$, $f(-3) = -270$, $f(-4) = -1152$

61. Yes; sample answer: Let $f(x)$ be the related polynomial function. Using the Factor Theorem, since $f(1) = 0$, $(x - 1)$ is a factor of the polynomial.

**Extension** Ask students to find the values of $a$, $b$, and $c$ so that when $x^6 - 2x^4 + ax^2 + bx + c$ is divided by $(x - 1)$, $(x - 2)$, and $(x + 4)$, the remainder is 0. −125, 342, −216
Chapter 2 Mid-Chapter Quiz

Lessons 2-1 to 2-3

Formative Assessment

Use the Mid-Chapter Quiz to assess students’ progress in the first half of the chapter.

For problems answered incorrectly, have students review the lessons indicated in parentheses.

Additional Answers

5a.

14. The degree is 4 and the leading coefficient is $-7$. Because the degree is even and the leading coefficient is negative, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$.

15. The degree is 5 and the leading coefficient is $-5$. Because the degree is odd and the leading coefficient is negative, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$.

16a. Sample answer: $f(x) = -2.707x^3 + 41.392x^2 - 141.452x + 238.176$

16b. Sample answer: $-177.273$ kWh; This answer does not make sense because it is not possible to consume negative kWh in a month.

23. yes; $f(x) = (x + 5)(x - 5)(x + 2)$

24. yes; $f(x) = (x - 1)(x - 2)(x - 4)(x + 1)$

5. TREES The heights of several fir trees and the areas under their branches are shown in the table. (Lesson 2-1)

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>Area (m^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>37.96</td>
</tr>
<tr>
<td>2.1</td>
<td>7.44</td>
</tr>
<tr>
<td>3.4</td>
<td>23.54</td>
</tr>
<tr>
<td>1.7</td>
<td>4.75</td>
</tr>
<tr>
<td>4.6</td>
<td>48.48</td>
</tr>
</tbody>
</table>

a. Create a scatter plot of the data. See margin. $y = 1.37x^2$

b. Predict the area under the branches of a fir tree that is $7.6$ meters high. $149.26$ m²

Solve each equation. (Lesson 2-1)

6. $\sqrt[3]{5x + 7} = 13$ $32.4$

7. $\sqrt[2] {2x - 2} + 1 = x$ $1, 3$


9. $-5 = (\sqrt[4]{x + 3})^2 - 32$ $13$

State the number of possible real zeros and turning points of each function. Then find all of the real zeros by factoring. (Lesson 2-2)

10. $f(x) = x^2 - 11x - 26$ $2$ and $13$

11. $f(x) = 3x^2 + 2x - x^3$ $11$ real zeros and 4 turning points; $-1, 0, 1, \frac{1}{3}$

12. $f(x) = x^4 + 9x^2 - 10$ $12$ real zeros and 3 turning points; $-1$ and $1$

13. MULTIPLE CHOICE Which of the following describes the possible end behavior of a polynomial of odd degree? (Lesson 2-2) D

A. $\lim_{x \to -\infty} f(x) = 5$, $\lim_{x \to \infty} f(x) = -5$

B. $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = -\infty$

C. $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$

D. $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. (Lesson 2-2)

14. $f(x) = -7x^4 - 3x^3 - 8x^2 + 23x + 7$ See margin.

15. $f(x) = -5x^2 + 4x^4 + 17x^2 - 8$ See margin.

16. ENERGY Crystal’s electricity consumption measured in kilowatt hours (kWh) for the past 12 months is shown below. (Lesson 2-2)

<table>
<thead>
<tr>
<th>Month</th>
<th>Consumption (kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>395</td>
</tr>
<tr>
<td>February</td>
<td>200</td>
</tr>
<tr>
<td>March</td>
<td>180</td>
</tr>
<tr>
<td>April</td>
<td>150</td>
</tr>
<tr>
<td>May</td>
<td>100</td>
</tr>
<tr>
<td>June</td>
<td>75</td>
</tr>
<tr>
<td>July</td>
<td>230</td>
</tr>
<tr>
<td>August</td>
<td>300</td>
</tr>
<tr>
<td>September</td>
<td>230</td>
</tr>
<tr>
<td>October</td>
<td>345</td>
</tr>
<tr>
<td>November</td>
<td>240</td>
</tr>
<tr>
<td>December</td>
<td>260</td>
</tr>
</tbody>
</table>

a. a-b. See margin.

b. Determine a model for the number of kilowatt hours Crystal will use the following January. Does this answer make sense? Explain your reasoning.

Divide using synthetic division. (Lesson 2-3)

17. $(2x^2 - 7x^2 + 8x - 3)(x - 1) - 5x^2 - 2x + 6 - \frac{7}{x - 1}$

18. $(x^3 - 9x^2 + 18) + (x - 1) - \frac{1}{x^2} + 2x - 5 + \frac{8}{x - 2}$

19. $(2x^2 + 11x^3 + 9x - 6)(x - 1) - \frac{5x^2 + 2}{x - 1}$

Determine each $f(x)$ using synthetic substitution. (Lesson 2-3)

20. $f(x) = x^2 + 4x^4 - 3x^2 + 18x^2 - 16x + 8$, $c = 2$ $376$

21. $f(x) = 6x^4 - 3x^5 + 8x^4 + 12x^2 - 6x + 4$, $c = -3$ $5881$

22. $f(x) = -2x^4 + 8x^5 - 12x^2 + 3x^3 - 8x^2 + 6x - 3$, $c = -2$ $-695$

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$. (Lesson 2-3) 23–24. See margin.

23. $f(x) = x^2 + 2x - 25 = 50$; $(x + 5)$

24. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$, $(x - 1), (x - 2)$

25. MULTIPLE CHOICE Find the remainder when $f(x) = x^3 - 4x + 5$ is divided by $x + 3$. (Lesson 2-3) F

A. $-10$  H. $20$

B. $8$  J. $26$
Zeros of Polynomial Functions

1 Real Zeros

Recall that a polynomial function of degree \( n \) can have at most \( n \) real zeros. These real zeros are either rational or irrational.

When a polynomial function has integer coefficients, and \( a_n \neq 0 \), then every rational zero of \( f(x) \) has the form \( \frac{p}{q} \), where

- \( p \) and \( q \) have no common factors other than \( \pm 1 \).
- \( p \) is an integer factor of the constant term \( a_n \).
- \( q \) is an integer factor of the leading coefficient \( a_n \).

**Corollary** If the leading coefficient \( a_n \) is 1, then any rational zeros of \( f(x) \) are integer factors of the constant term \( a_n \).

Once you know all of the possible rational zeros of a polynomial function, you can then use direct or synthetic substitution to determine which, if any, are actual zeros of the polynomial.

**Example 1** Leading Coefficient Equal to 1

List all possible rational zeros of each function. Then determine which, if any, are zeros.

a. \( f(x) = x^2 + 2x + 1 \)

**Step 1** Identify possible rational zeros.

Because the leading coefficient is 1, the possible rational zeros are the integer factors of the constant term 1. Therefore, the possible rational zeros of \( f(x) \) are 1 and \(-1\).

**Step 2** Use direct substitution to test each possible zero.

\[ f(1) = (1)^2 + 2(1) + 1 = 4 \]
\[ f(-1) = (-1)^2 + 2(-1) + 1 = 0 \]

Because \( f(1) \neq 0 \) and \( f(-1) \neq 0 \), you can conclude that \( f(x) \) has no rational zeros. From the graph of \( f(x) \), you can see that \( f(x) \) has one real zero. Applying the Rational Zeros Theorem shows that this zero is irrational.
How can this equation be solved? Set it equal to 0: \(-0.0007x^2 + 2.45x - 1500 = 0\). Then factor or use the Quadratic Formula.

What is the greatest number of real zeros this function can have? How do you know? 2; The degree is 2.

1 Real Zeros

Examples 1–3 show how to find all rational zeros of a polynomial function. Example 4 shows how to use upper and lower bounds to help find real zeros of a polynomial. Example 5 uses Descartes’ Rule of Signs to help find real zeros of a polynomial.

Formative Assessment

Use the Guided Practice exercises after each example to determine students’ understanding of concepts.

Additional Examples

1 List all possible rational zeros of each function. Then determine which, if any, are zeros.
   a. \(f(x) = x^3 - 3x^2 - 2x + 4\)
      \(\pm 1, \pm 2, \pm 4, 1\)
   b. \(f(x) = x^3 - 2x - 1\)
      \(\pm 1, -1\)

2 List all possible rational zeros of \(f(x) = 2x^3 - 5x^2 - 28x + 15\). Then determine which, if any, are zeros.
   \(\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}, \pm \frac{1}{2}, -3, 5\)

Additional Examples also in Interactive Classroom PowerPoint® Presentations

b. \(g(x) = x^4 + 4x^3 - 12x - 9\)
   **Step 1**
   Because the leading coefficient is 1, the possible rational zeros are the integer factors of the constant term -9. Therefore, the possible rational zeros of \(g\) are \(\pm 1, \pm 3, \pm 9\).
   **Step 2**
   Begin by testing 1 and \(-1\) using synthetic substitution.
   
<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>4</th>
<th>0</th>
<th>-12</th>
<th>-9</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>3</td>
<td>3</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>
   
   Because \(g(-1) = 0\), you can conclude that \(-1\) is a zero of \(g\). Testing 3 on the depressed polynomial shows that \(-3\) is another rational zero.

Thus, \(g(x) = (x + 1)(x + 3)(x^2 - 3)\). Because the factor \((x^2 - 3)\) yields no rational zeros, we can conclude that \(g\) has only two rational zeros, \(-1\) and \(-3\).

Guided Practice

List all possible rational zeros of each function. Then determine which, if any, are zeros.
1A. \(f(x) = x^3 + 5x^2 - 4x - 2\)
   1B. \(h(x) = x^4 + 3x^3 - 7x^2 + 9x - 30\)

When the leading coefficient of a polynomial function is not 1, the list of possible rational zeros can increase significantly.

Example 2 Leading Coefficient not Equal to 1

List all possible rational zeros of \(h(x) = 3x^4 - 7x^2 + 8x + 8\). Then determine which, if any, are zeros.

**Step 1**

The leading coefficient is 3 and the constant term is 8.

Possible rational zeros: Factors of 8 = \(\pm 1, \pm 2, \pm 4, \pm 8\) or \(\pm 1, \pm 2, \pm 4, \pm 8\) or \(\pm 1, \pm 2, \pm 4, \pm 8\)

**Step 2**

By synthetic substitution, you can determine that \(-2\) is a rational zero.

\[
\begin{array}{c|cccc}
  -2 & 3 & -7 & -22 & 8 \\
  \hline
  1 & 1 & 3 & 26 & 8 \\
  | 1 & 1 & 4 & 0 |
\end{array}
\]

By the division algorithm, \(h(x) = (x + 2)(3x^3 - 13x + 4)\). Once \(3x^3 - 13x + 4\) is factored, the polynomial becomes \(h(x) = (x + 2)(3x - 1)(x - 4)\), and you can conclude that the rational zeros of \(h\) are \(-2, \frac{1}{3}, 4\). Check this result by graphing.

Guided Practice

List all possible rational zeros of each function. Then determine which, if any, are zeros.
2A. \(g(x) = 2x^3 - 4x^2 + 18x - 36\)
   2B. \(f(x) = 3x^4 - 18x^3 + 2x - 21\)
**Guided Practice**

3. **Volleyball** A volleyball that is returned after a serve with an initial speed of 40 feet per second at a height of 4 feet is given by \( f(t) = 4 + 40t - 16t^2 \), where \( f(t) \) is the height the ball reaches in feet and \( t \) is time in seconds. At what time(s) will the ball reach a height of 20 feet?

One way to narrow the search for real zeros is to determine an interval within which all real zeros of a function are located. A real number \( a \) is a lower bound for the real zeros of \( f \) if \( f(x) \neq 0 \) for \( x < a \). Similarly, \( b \) is an upper bound for the real zeros of \( f \) if \( f(x) \neq 0 \) for \( x > b \).

You can test whether a given interval contains all real zeros of a function by using the following upper and lower bound tests.

### Key Concept: Upper and Lower Bound Tests

Let \( f \) be a polynomial function of degree \( n \geq 1 \), real coefficients, and a positive leading coefficient. Suppose \( f(x) \) is divided by \( x - c \) using synthetic division.

- If \( c \leq 0 \) and every number in the last line of the division is alternately nonnegative and nonpositive, then \( c \) is a lower bound for the real zeros of \( f \).
- If \( c \geq 0 \) and every number in the last line of the division is nonnegative, then \( c \) is an upper bound for the real zeros of \( f \).

---

**Real-World Example 3: Solve a Polynomial Equation**

**Business** After the first half-hour, the number of video games that were sold by a company on their release date can be modeled by \( g(x) = 2x^3 + 4x^2 - 2x \), where \( g(x) \) is the number of games sold in hundreds and \( x \) is the number of hours after the release. How long did it take to sell 400 games?

Because \( g(x) \) represents the number of games sold in hundreds, you need to solve \( g(x) = 4 \) to determine how long it will take to sell 400 games.

\[
g(x) = 4
\]

Write the equation.

\[
2x^3 + 4x^2 - 2x = 4
\]

Substitute \( 2x^3 + 4x^2 - 2x \) for \( g(x) \).

\[
2x^3 + 4x^2 - 2x - 4 = 0
\]

Subtract 4 from each side.

Apply the Rational Zeros Theorem to this new polynomial function, \( f(x) = 2x^3 + 4x^2 - 2x - 4 \).

**Step 1**  Possible rational zeros: Factors of 4

\[
\pm 1, \pm 2, \pm 4
\]

Factors of 2

\[
\pm 1, \pm 2
\]

Possible rational zeros:

\[
= \pm 1, \pm 2, \pm 4, \pm \frac{1}{2}
\]

**Step 2**  By synthetic substitution, you can determine that 1 is a rational zero.

\[
\begin{array}{c|ccccc}
1 & 2 & 4 & -2 & -4 & 0 \\
 & 2 & 6 & 4 & 0 & \\
\end{array}
\]

Because 1 is a zero of \( f \), \( x = 1 \) is a solution of \( f(x) = 0 \). The depressed polynomial \( 2x^2 + 6x + 4 \) can be written as \( 2(x + 2)(x + 1) \). The zeros of this polynomial are \(-2 \) and \(-1\). Because time cannot be negative, the solution is \( x = 1 \). So, it took 1 hour to sell 400 games.

**Real-World Link**

A recent study showed that almost a third of frequent video game players are between 6 and 17 years old.

Source: NPD Group Inc
Focus on Mathematical Content
Zeros  An upper bound for the zeros of a polynomial function is a number for which no real zero greater than that number exists for that function. A lower bound for the zeros of a polynomial function is a number for which no real zero less than that number exists. The upper and lower bounds chosen are not unique.

Testing the chosen bounds is done by synthetically dividing the polynomial by the linear expressions \( x - c \), where \( c \) is each number chosen. If every number in the last line of the division is nonnegative, the divisor is an upper bound. If every number in the last line of the division is nonnegative and nonpositive, the divisor is a lower bound. If every number in the last line of the division is nonnegative, the divisor is an upper bound. Finding upper and lower bounds can help eliminate values from the list of possible zeros found when using the Rational Zero Theorem.

Additional Example
4 Determine an interval in which all real zeros of \( f(x) = x^4 - 4x^3 - 11x^2 - 4x - 12 \) must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros. Upper and lower bounds may vary. Sample answer: \([-3, 7]\); With synthetic division, the values alternate signs when testing \(-3\), and are all nonnegative when testing \(7\). So, \(-3\) is a lower bound and \(7\) is an upper bound. The zeros are \(-2\) and \(6\).

Example 4 Use the Upper and Lower Bound Tests
Determine an interval in which all real zeros of \( h(x) = 2x^4 - 11x^3 + 2x^2 - 44x - 24 \) must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros.

Step 1 Graph \( h(x) \) using a graphing calculator. From this graph, it appears that the real zeros of this function lie in the interval \([-1, 7]\).

Step 2 Test a lower bound of \( c = -1 \) and an upper bound of \( c = 7 \).

\[
\begin{array}{cccc}
-1 & 2 & -11 & 2 & -44 & -24 \\
 & -2 & 13 & -15 & -59 & \\
 & 2 & -13 & 15 & -59 & 35
\end{array}
\]

Values alternate signs in the last line, so \(-1\) is a lower bound.

\[
\begin{array}{cccc}
7 & 2 & -11 & 2 & -44 & -24 \\
 & 14 & 21 & 161 & 819 & \\
 & 2 & 3 & 23 & 117 & 795
\end{array}
\]

Values are all nonnegative in the last line, so \(7\) is an upper bound.

Step 3 Use the Rational Zero Theorem.
Possible rational zeros: \( \pm \frac{1}{2}, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24 \)

\[
= \pm 1, \pm 2, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm \frac{1}{2}, \pm \frac{3}{2}
\]

Because the real zeros are in the interval \([-1, 7]\), you can narrow this list to just \(\pm 1, \pm \frac{1}{2}, \pm \frac{3}{2}, 2, 4, \) or 6. From the graph, it appears that only 6 and \(-\frac{1}{2}\) are reasonable.

Begin by testing 6.

\[
\begin{array}{cccc}
6 & 2 & -11 & 2 & -44 & -24 \\
 & 12 & 6 & 48 & 24 & \\
 & 2 & 1 & 8 & 4 & 0
\end{array}
\]

By the division algorithm, \( h(x) = 2(x - 6)\left(x + \frac{1}{2}\right)(x^2 + 4) \). Notice that the factor \((x^2 + 4)\) has no real zeros associated with it because \(x^2 + 4\) has no real solutions. So, \(x\) has two real solutions that are both rational, 6 and \(-\frac{1}{2}\). The graph of \( h(x) = 2x^4 - 11x^3 + 2x^2 - 44x - 24 \) supports this conclusion.

Guided Practice
Determine an interval in which all real zeros of the given function must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros.

4A. \( g(x) = 6x^4 + 70x^3 - 21x^2 + 35x - 12 \)

Sample answer: \([-13, 1]; -12, \frac{1}{3}\)

4B. \( f(x) = 10x^5 - 50x^4 - 3x^3 + 22x^2 - 4x + 30 \)

Sample answer: \([-2, 6]; 5, 0.71, -1.16\)
Another way to narrow the search for real zeros is to use **Descartes’ Rule of Signs**. This rule gives us information about the number of positive and negative real zeros of a polynomial function by looking at a polynomial’s variations in sign.

**Key Concept: Descartes’ Rule of Signs**

If \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) is a polynomial function with real coefficients, then

- the number of positive real zeros of \( f(x) \) is equal to the number of variations in sign of \( f(x) \) or less than that number by some even number and
- the number of negative real zeros of \( f(x) \) is the same as the number of variations in sign of \( f(-x) \) or less than that number by some even number.

**Example 5: Use Descartes’ Rule of Signs**

Describe the possible real zeros of \( g(x) = -3x^3 + 2x^2 - x - 1 \).

Examine the variations in sign for \( g(x) \) and for \( g(-x) \).

\[
\begin{align*}
g(x) &= -3x^3 + 2x^2 - x - 1 \\
g(-x) &= -3(-x)^3 + 2(-x)^2 - (-x) - 1 \\
&= 3x^3 + 2x^2 + x - 1
\end{align*}
\]

The original function \( g(x) \) has two variations in sign, while \( g(-x) \) has one variation in sign. By Descartes’ Rule of Signs, you know that \( g(x) \) has either 2 or 0 positive real zeros and 1 negative real zero.

From the graph of \( g(x) \) shown, you can see that the function has one negative real zero close to \( x = -0.5 \) and no positive real zeros.

**Guided Practice**

Describe the possible real zeros of each function.

5A. \( h(x) = 6x^2 + 8x^2 - 10x - 15 \)
5B. \( f(x) = -11x^4 + 20x^3 + 3x^2 - x + 18 \)

5A. 1 positive real zero, 2 or 0 negative real zeros
5B. 3 or 1 positive real zeros, 1 negative real zero

When using Descartes’ Rule of Signs, the number of real zeros indicated includes any repeated zeros. Therefore, a zero with multiplicity \( m \) should be counted as \( m \) zeros.

**2 Complex Zeros**

Just as quadratic functions can have real or imaginary zeros, polynomials of higher degree can also have zeros in the complex number system. This fact, combined with the **Fundamental Theorem of Algebra**, allows us to improve our statement about the number of zeros for any \( n \)-th-degree polynomial.

**Key Concept: Fundamental Theorem of Algebra**

A polynomial function of degree \( n \), where \( n > 0 \), has at least one zero (real or imaginary) in the complex number system.

**Corollary**: A polynomial function of degree \( n \) has exactly \( n \) zeros, including repeated zeros, in the complex number system.
Tips for New Teachers

- **$i^2$** Remind students that $i^2 = -1$, and therefore, $-i^2 = -(−1)$ or $1$.

Focus on Mathematical Content

Existence Theorems The Fundamental Theorem of Algebra and the Linear Factorization Theorems are called **existence theorems**. They tell you that the zeros or linear factors of a polynomial exist, but not how to find them.

Key Concept Linear Factorization Theorem

If $f(x)$ is a polynomial function of degree $n > 0$, then $f$ has exactly $n$ linear factors and

\[ f(x) = a_n(x - c_1)(x - c_2) \ldots (x - c_n) \]

where $a_n$ is some nonzero real number and $c_1, c_2, \ldots, c_n$ are the complex zeros (including repeated zeros) of $f$.

According to the **Conjugate Root Theorem**, when a polynomial equation in one variable with real coefficients has a root of the form $a + bi$, where $b \neq 0$, then its complex conjugate, $a - bi$, is also a root. You can use this theorem to write a polynomial function given its complex zeros.

Example 6 Find a Polynomial Function Given Its Zeros

Write a polynomial function of least degree with real coefficients in standard form that has $-2, 4, 3 - i$, and $3 + i$ as zeros.

Because $3 - i$ is a zero and the polynomial is to have real coefficients, you know that $3 + i$ must also be a zero. Using the Linear Factorization Theorem and the zeros $-2, 4, 3 - i$, and $3 + i$, you can write $f(x)$ as follows.

\[ f(x) = a(x - (-2))(x - 4)(x - (3 - i))(x - (3 + i)) \]

While $a$ can be any nonzero real number, it is simplest to let $a = 1$. Then write the function in standard form.

\[ f(x) = (x + 2)(x - 4)(x - (3 - i))(x - (3 + i)) \]

\[ f(x) = (x^2 - 2x - 8)(x^2 - 6x + 10) \]

\[ f(x) = x^4 - 8x^3 + 14x^2 + 28x - 80 \]

Therefore, a function of least degree that has $-2, 4, 3 - i$, and $3 + i$ as zeros is $f(x) = x^4 - 8x^3 + 14x^2 + 28x - 80$ or any nonzero multiple of $f(x)$.

Guided Practice

Write a polynomial function of least degree with real coefficients in standard form that has the given zeros. **Sample answers given.**

6A. $-3, 1$ (multiplicity: 2), $4i$

6B. $2\sqrt{3}, -2\sqrt{3}, 1 + i$

6A. $f(x) = x^3 + x^4 + 11x^3 + 19x^2 - 80x + 48$

6B. $f(x) = x^4 - 2x^3 - 10x^2 + 24x - 24$

In Example 6, you wrote a function with real and complex zeros. A function has complex zeros when its factored form contains a quadratic factor which is irreducible over the reals. A quadratic expression is **irreducible over the reals** when it has real coefficients but no real zeros associated with it. This example illustrates the following theorem.

Key Concept Factoring Polynomial Functions Over the Reals

Every polynomial function of degree $n > 0$ with real coefficients can be written as the product of linear factors and irreducible quadratic factors, each with real coefficients.

As indicated by the Linear Factorization Theorem, when factoring a polynomial function over the complex number system, we can write the function as the product of only linear factors.

Logical Learners Ask students to compare the products for $(x - 3)(x - 2)(x - 1)$ and $(x + 3)(x + 2)(x + 1)$. Encourage students to notice that one of these polynomials has only positive roots while the other has only negative roots. Then pose the following question: What features of the polynomials themselves give us a clue to this fact? The second product has all positive coefficients, while the first has alternating signs.
Example 7: Factor and Find the Zeros of a Polynomial Function

Consider \( k(x) = x^3 - 18x^2 + 30x^2 - 19x + 30 \).

a. Write \( k(x) \) as the product of linear and irreducible quadratic factors.

The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 6, \pm 10, \pm 15, \pm 30 \). The original polynomial has 4 sign variations.

\[ k(-x) = (-x)^3 - 18(-x)^2 + 30(-x)^2 - 19(-x) + 30 \]

\[ = -x^3 + 18x^2 + 30x^2 + 19x + 30 \]

\( k(-x) \) has 1 sign variation, so \( k(x) \) has 4, 2, or 0 positive real zeros and 1 negative real zero.

The graph shown suggests \(-5\) as one real zero of \( k(x) \). Use synthetic substitution to test this possibility.

\[
\begin{array}{c|cccc}
-5 & 1 & -3 & 30 & 30 \\
\hline
1 & 1 & -3 & 0 & 0 \\
\end{array}
\]

Because \( k(x) \) has only 1 negative real zero, you do not need to test any other possible negative rational zeros. Zooming in on the positive real zeros in the graph suggests 2 and 3 as other rational zeros. Test these possibilities successively in the depressed quartic and then cubic polynomials.

\[
\begin{array}{c|cccc}
2 & 1 & 0 & -18 & 30 \\
\hline
1 & 1 & 25 & 25 & -30 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
3 & 1 & -3 & 1 & -3 \\
\hline
1 & 0 & 3 & 0 & 0 \\
\end{array}
\]

The remaining quadratic factor \((x^2 + 1)\) yields no real zeros and is therefore irreducible over the reals. So, \( k(x) \) written as a product of linear and irreducible quadratic factors is \( k(x) = (x + 5)(x - 2)(x - 3)(x^2 + 1) \).

b. Write \( k(x) \) as the product of linear factors.

You can factor \( x^2 + 1 \) by writing the expression first as a difference of squares \((\sqrt{-1})^2\) or \(x^2 - i^2\). Then factor this difference of squares as \((x - i)(x + i)\). So, \( k(x) \) written as a product of linear factors is as follows.

\[ k(x) = (x + 5)(x - 2)(x - 3)(x - i)(x + i) \]

c. List all the zeros of \( k(x) \).

Because the function has degree 5, by the corollary to the Fundamental Theorem of Algebra \( k(x) \) has exactly five zeros, including any that may be repeated. The linear factorization gives us these five zeros: \(-5, 2, 3, i, -i\).

Guided Practice

Write each function as (a) the product of linear and irreducible quadratic factors and (b) the product of linear factors. Then (c) list all of its zeros.

7A. \( f(x) = (x^2 + 4) \cdot (x - 5)(x + 6) \)

b. \( f(x) = (x - 2i) \cdot (x + 2i)(x - 5i) \cdot (x + 6) \)

3. \( \pm 2i, 5, -6 \)

7B. \( a. f(x) = (x^2 + 9) \cdot (x - 1)(x - 4) \cdot (x + 3) \)

b. \( f(x) = (x - 3)(x + 3)(x - 1) \cdot (x - 4)(x + 3) \cdot \pm 3i, 1, 4, -3 \)

c. \( f(x) = x^3 + 26x^2 + 4x - 120 \)

b. \( f(x) = x^3 - 2x^2 - 6x^2 - 9x + 108 \)
Find all complex zeros of 
\( p(x) = x^4 - 6x^3 + 35x^2 - 50x - 58 \) given that 2 + 5i is a zero of \( p \). Then write the linear factorization of \( p(x) \).

\[ z = 2 + 5i \]

\[ p(x) = [x - (2 + 5i)] [x - (2 - 5i)] [x - (1 + \sqrt{3})] [x - (1 - \sqrt{3})] \]

Find the Zeros of a Polynomial When One is Known

Find all complex zeros of \( p(x) = x^4 - 6x^3 + 20x^2 - 22x + 13 \) given that 2 + 3i is a zero of \( p \). Then write the linear factorization of \( p(x) \).

Use synthetic substitution to verify that 2 + 3i is a zero of \( p \).

\[ p(x) = x^4 - 6x^3 + 20x^2 - 22x + 13 \]

\[ 2 + 3i \]

Therefore, the four zeros of \( p(x) \) are 2 + 3i, 2 - 3i, 1 + \( \sqrt{3} \), and 1 - \( \sqrt{3} \). The linear factorization of \( p(x) \) is \( \text{above} \) 2 + 3i.

Using the graph of \( p(x) \), you can verify that the function has two real zeros at 1 + \( \sqrt{2} \) or about 2.4 and 1 - \( \sqrt{2} \) or about -0.41.

Guided Practice

For each function, use the given zero to find all the complex zeros of the function. Then write the linear factorization of the function.

8A. \( g(x) = x^4 - 10x^3 + 35x^2 - 66x + 10 \)

8B. \( h(x) = x^4 - 8x^3 + 26x^2 - 8x + 26 \)

19. Sample answer: \([-6, 5] \]

20. Sample answer: \([-6, 4] \]

21. Sample answer: \([-4, 7] \]

22. Sample answer: \([-2, 9] \]

23. Sample answer: \([-2, 7] \]

24. Sample answer: \([-3, 5] \]

25. Sample answer: \([-3, 5] \]

26. 1 positive zero, 2 or 0 negative zeros

27. 3 or 1 positive zeros, 1 negative zero

28. 2 or 0 positive zeros, 2 or 0 negative zeros

29. 2 or 0 positive zeros, 2 or 0 negative zeros

30. 3 or 1 positive zeros, 2 or 0 negative zeros

31. 4, 2 or 0 positive zeros, 1 negative zero
Exercises

List all possible rational zeros of each function. Then determine which, if any, are zeros. (Examples 1 and 2)

1. \( g(x) = x^4 - 6x^3 - 31x^2 + 216x - 180 \) 1–8. See margin.
2. \( f(x) = 4x^3 - 24x^2 - x + 6 \)
3. \( g(x) = x^4 - x^3 - 31x^2 + x + 30 \)
4. \( g(x) = -4x^4 + 35x^3 - 87x^2 + 56x + 20 \)
5. \( h(x) = 6x^4 + 13x^3 - 67x^2 - 156x - 60 \)
6. \( f(x) = 3x^4 + 12x^3 + 56x^2 + 48x - 64 \)
7. \( f(x) = x^4 - 11x^3 + 49x^2 - 147x^2 + 360x - 432 \)
8. \( g(x) = 8x^3 + 18x^4 - 5x^2 - 72x^2 - 162x + 45 \)

9. MANUFACTURING The specifications for the dimensions of a new cardboard container are shown. If the volume of the container is modeled by \( V(h) = 2h^3 - 9h^2 + 4h \) and it will hold 45 cubic inches of merchandise, what are the container’s dimensions? (Example 3)

Solve each equation. (Example 3)

10. \( x^4 + 2x^3 - 7x^2 - 20x - 12 = 0 \)
11. \( x^4 + 9x^3 + 23x^2 + 3x - 36 = 0 \)
12. \( x^4 - 2x^3 - 7x^2 + 8x + 12 = 0 \)
13. \( x^4 - 3x^3 - 20x^2 + 84x - 80 = 0 \)
14. \( x^4 + 34x^3 + 6x^2 + 21x^2 - 48 - 3x^2 - 2 = -1 \)
15. \( 16x^4 + 3x^3 - 4x^2 - 2x + 1 \)
16. \( x^4 + 3x^3 + 3x^2 - 2x - 1 \)

17. SALES The sales \( S(x) \) in thousands of dollars of a store makes during one month can be approximated by \( S(x) = 2x^2 - 2x^3 + 4x \), where \( x \) is the number of days after the first day of the month. How many days will it take the store to make \( $16,000 \) ? (Example 3) 2 days

Determine an interval in which all real zeros of each function must lie. Explain your reasoning using the upper and lower bound tests. Then find all the real zeros. (Example 4)

18. \( f(x) = x^4 - 9x^3 + 12x^2 + 44x - 48 \)
19. \( f(x) = 2x^4 - x^3 - 29x^2 + 34x + 24 \)
20. \( g(x) = 2x^4 + 4x^3 - 18x^2 - 4x + 16 \)
21. \( g(x) = 6x^4 - 33x^3 - 6x^2 + 123x - 90 \)
22. \( f(x) = 2x^4 - 17x^3 + 39x^2 - 16x - 20 \)
23. \( f(x) = 2x^4 - 13x^3 + 21x^2 + 9x - 27 \)
24. \( h(x) = x^4 - x^3 - 9x^2 + 5x^2 + 16x - 12 \)
25. \( h(x) = 4x^4 - 20x^3 + 5x^2 + 80x^2 - 75x + 18 \)

Describe the possible real zeros of each function. (Example 5)

26. \( f(x) = -2x^3 - 3x^2 + 4x + 7 \)
27. \( f(x) = 10x^4 - 3x^2 + 8x + 4x - 8 \)
28. \( f(x) = -3x^3 + 5x^2 + 4x^2 - 2x - 6 \)
29. \( f(x) = 12x^4 - 6x^3 + 3x^2 - 2x - 12 \)
30. \( g(x) = 4x^5 + 3x^4 + 9x^3 - 8x^2 + 16x - 24 \)
31. \( h(x) = -4x^5 + x^4 - 3x^3 - 24x^2 - 64x - 124 \)

Write a polynomial function of least degree with real coefficients in standard form that has the given zeros. (Example 6) 32–41. See margin.

32. \( -3, -4, 6, -1 \)
33. \( -2, -4, -3, 5 \)
34. \( -5, 3, 4 + i \)
35. \( -1, 8, 6 - i \)
36. \( -3, 2, -2, 1 \)
37. \( -5, 2, 4 - \sqrt{5}, 4 + \sqrt{3} \)
38. \( \sqrt{7}, -\sqrt{7}, 4i \)
39. \( \sqrt{6}, -\sqrt{6}, 3 - 4i \)
40. \( 2 + \sqrt{5}, 2 - \sqrt{5}, 4 + 5i \)
41. \( 6 - \sqrt{5}, 6 + \sqrt{5}, 8 - 3i \)

Write each function as (a) the product of linear and irreducible quadratic factors and (b) the product of linear factors. Then (c) list all of its zeros. (Example 7)

42. \( g(x) = x^4 - 3x^3 - 12x^2 + 20x + 48 \)
43. \( g(x) = x^4 - 3x^3 - 12x^2 + 8 \)
44. \( h(x) = x^4 + 2x^3 + 15x^2 + 18x - 216 \)
45. \( f(x) = 4x^4 - 35x^3 + 140x^2 - 295x + 156 \)
46. \( f(x) = 4x^4 - 15x^3 + 43x^2 + 577x + 615 \)
47. \( h(x) = x^2 - 2x^3 - 17x^2 + 4x + 30 \)
48. \( g(x) = x^4 + 31x^2 - 180 \)
49–54. See Chapter 2 Answer Appendix.

Use the given zero to find all complex zeros of each function. Then write the linear factorization of the function. (Example 8)

49. \( h(x) = 2x^4 + x^3 + 7x^2 + 21x^2 - 225x + 108 ; 3 \)
50. \( h(x) = 3x^5 - 5x^3 - 13x^5 - 65x^2 - 2200x + 1500 ; -5i \)
51. \( g(x) = x^5 - 2x^4 + 13x^3 + 28x^2 + 46x - 60 ; 3 - i \)
52. \( g(x) = 4x^5 - 57x^4 + 287x^3 - 547x^2 + 83x + 510 ; 4 + i \)
53. \( g(x) = x^5 - 3x^4 - 4x^3 - 12x^2 - 32x + 96 ; -2i \)
54. \( f(x) = x^4 - 10x^3 + 35x^2 - 46x + 10 ; 3 + i \)

55. ARCHITECTURE An architect is constructing a scale model of a building that is in the shape of a pyramid.

a. If the height of the scale model is 9 inches less than its length and its base is a square, write a polynomial function that describes the volume of the model in terms of its length. a–c. See margin.

b. If the volume of the model is 6300 cubic inches, write an equation describing the situation.

c. What are the dimensions of the scale model?

Additional Answers

32. Sample answer: \( f(x) = x^4 - 4x^3 - 23x^2 + 54x + 72 \)
33. Sample answer: \( f(x) = x^4 + 4x^3 - 19x^2 - 106x - 120 \)
34. Sample answer: \( f(x) = x^4 - 6x^3 - 14x^2 + 154x - 255 \)
35. Sample answer: \( f(x) = x^4 - 19x^2 + 113x^2 - 163x - 296 \)
36. Sample answer: \( f(x) = x^4 - 4x^3 - 41x^2 + 80x + 420 \)
37. Sample answer: \( f(x) = x^4 - 5x^3 - 21x^2 + 119x - 130 \)
38. Sample answer: \( f(x) = x^4 + 9x^2 - 112 \)
39. Sample answer: \( f(x) = x^4 - 6x^3 + 19x^2 + 36x - 150 \)
40. Sample answer: \( f(x) = x^4 - 12x^3 + 74x^2 - 172x + 41 \)
41. Sample answer: \( f(x) = x^4 - 28x^3 + 296x^2 - 1372x + 2263 \)

55a. \( V(\ell) = \frac{1}{3} \ell^3 - 3 \ell^2 \)
55b. \( 6300 = \frac{1}{3} \ell^3 - 3 \ell^2 \)
55c. base: 30 in. by 30 in.; height: 21 in.
56. **CONSTRUCTION** The height of a tunnel that is under construction is 1 foot more than half its width and its length is 32 feet more than 324 times its width. If the volume of the tunnel is 62,231,040 cubic feet and it is a rectangular prism, find the length, width, and height.

\[ \ell = 23,360 \text{ ft}, w = 72 \text{ ft}, h = 37 \text{ ft} \]

Write a polynomial function of least degree with integer coefficients that has the given number as a zero.

57. \( \sqrt{6} \)

58. \( \sqrt[3]{6} \)

59. \(-\sqrt{2} \)

60. \(-\sqrt[3]{2} \)

61. \( g(x) = 3x^4 - 15x^3 + 87x^2 - 375x + 300 \)

\( g(x) = 3(x - 4) \cdot (x - 1) \cdot (x - 5) \cdot (x + 5); 4, 1, \pm 5i \)

62. \( g(x) = 2x^3 + 2x^2 + 28x^2 + 32x^2 - 64x \)

\( g(x) = 2x(x - 1) \cdot (x + 2) \cdot (x - 4); 0, 1, -2, \pm 4i \)

**H.O.T. Problems**

63. \( h(x) = 6x^4 - 6x^2 + 12 - 1 \)

64. \( f(y) = \frac{1}{4}y^4 + \frac{3}{2}y^3 - 2y + 8 \)

65. \( w(z) = \frac{1}{2}z^4 - 10z^3 + 30z^2 - 10z + 29 \)

66. \( b(x) = x^3 + 2x^2 + 3x^2 - 2x^3 - 3x + 1 \)

**ENRICHMENT**

A steel beam is supported by two pilings 200 feet apart. If a weight is placed x feet from the piling on the left, a vertical deflection represented by \( d = 0.000008x(200 - x) \) occurs. How far is the weight from the piling if the vertical deflection is 0.8 feet?

---

**Headache of the Day**

67. **MULTIPLE REPRESENTATIONS** In this problem, you will explore even- and odd-degree polynomial functions.

a. **ANALYTICAL** Identify the degree and number of zeros of each polynomial function.

\[ f(x) = x^3 - x^2 + 9x - 9 \quad 3, 3 \text{ zeros} \]

\[ g(x) = 2x^3 + x^2 - 13x + 6 \quad 3, 3 \text{ zeros} \]

\[ h(x) = 5x^3 + 2x^2 - 13x + 6 \quad 3, 3 \text{ zeros} \]

\[ i(x) = x^4 + 2x^3 + 144 \quad 4, 4 \text{ zeros} \]

\[ j(x) = 3x^4 + 5x^3 + 46x^2 - 32x^2 \quad 6, 6 \text{ zeros} \]

\[ k(x) = 4x^4 - 11x^3 + 10x^2 - 11x + 6 \quad 4, 4 \text{ zeros} \]

b. **GRAPHICAL** Find the zeros of each function.

c. **VERBAL** Does an odd-degree function have to have a minimum number of real zeros? Explain.

---

**Enrichment**

57. Sample answer: \( f(x) = x^3 - 6 \)

58. Sample answer: \( f(x) = x^3 - 5 \)

59. Sample answer: \( f(x) = x^3 + 2 \)

60. Sample answer: \( f(x) = x^3 + 7 \)

61. \( f(x) = x - 3 \sqrt[3]{3}x^2 + x \sqrt[3]{3} + \sqrt[3]{9} \)

62. \( f(x) = x + 2\sqrt[4]{2}(x^2 - 2x\sqrt[4]{2} + 4\sqrt{2}) \)

63. \( f(x) = (2x + \sqrt[3]{9})(4x^2 - 2x\sqrt[3]{9} + \sqrt[3]{81}) \)

64. \( f(x) = (3x^2 + \sqrt[4]{2})(9x^2 - 3x^2\sqrt[4]{2} + 2\sqrt{2}) \)

---

**Additional Answers**

67. **ERROR ANALYSIS** Angle and Julia are using the Rational Zeros Theorem to find all the possible rational zeros of \( f(x) = x^2 + 2x - 5x - 3 \). Angle thinks the possible zeros are \( \frac{1}{2}, \pm \frac{1}{3}, \pm 1, \pm 3 \) and Julia thinks they are \( \pm \frac{1}{2}, \pm \frac{1}{3}, \pm 1 \). Is either of them correct? Explain your reasoning.

**REASONING**

68. **OPEN ENDED** Write a function of 4th degree with an imaginary zero and an irrational zero. 

**Sample answer:** \( f(x) = x^4 - 2x^2 - 3 \)

69. **REASONING** Determine whether the statement is true or false. If false, provide a counterexample.

A third-degree polynomial with real coefficients has at least one nonzero real zero. 

**See Chapter 2 Answer Appendix.**

**CHALLENGE** Find the zeros of each function if \( h(x) \) has zeros at \( x_1, x_2, x_3, \) and \( x_4 \).

70. \( c(x) = 37h(x) \)

71. \( k(x) = h(3x) \)

72. \( g(x) = h(-x) \)

73. \( f(x) = h(x) \)

74. **REASONING** If \( x = c \) is a factor of \( f(x) = ax^3 + bx^2 + c \), what value must \( c \) be greater than or equal to in order to be an upper bound for the zeros of \( f(x) \)? Assume \( a > 0 \). Explain your reasoning.

**See Chapter 2 Answer Appendix.**

**WRITING IN MATH** Explain why a polynomial with real coefficients and one imaginary zero must have at least two imaginary zeros. 

**See Chapter 2 Answer Appendix.**

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128 | Lesson 2-4 | Zeros of Polynomial Functions
### Spiral Review

84. \((x^3 - 9x^2 + 27x - 28) ÷ (x - 3)\)  
85. \((x^4 + 9x - 1) ÷ (x - 2)\)  
86. \((3x^4 - 2x^3 + 5x^2 - 4x - 2) ÷ (x + 1)\)

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. (Lesson 2-3)

88–90. See margin.

88. \(f(x) = -4x^3 + 3x^2 + 6\)  
89. \(f(x) = 4x^6 + 2x^3 + 7x^2\)

### Skills Review for Standardized Tests

91. \(f(x) = (x - 1)(x + 10x + 3)\)  
92. \(f(x) = x(x - 2)(x + 21x - 4)\)  
93. \(f(x) = x^2(x - 3)(x + 4)\)

### Error Analysis

For Exercise 73, remind students that the first step in finding all the possible zeros is to write the polynomial function in standard form. Then identify the factors of the leading coefficient and the factors of the constant term.

### 4 Assess

#### Yesterday’s News

Ask students to write how the synthetic division teachings in Lesson 2-3 helped them test the possible rational zeros of a polynomial function.

#### Additional Answers

73. Julius; sample answer: Angie did not divide by the factors of the leading coefficient.

74. Sample answer: An input of \(-1\) produces a negative output, whereas an input of 0 results in a positive output.

88. The degree is 7, and the leading coefficient is \(-4\). Because the degree is odd and the leading coefficient is negative, \(\lim_{x \to -\infty} f(x) = -\infty\) and \(\lim_{x \to \infty} f(x) = \infty\).

89. The degree is 6, and the leading coefficient is 4. Because the degree is even and the leading coefficient is positive, \(\lim_{x \to -\infty} f(x) = \infty\) and \(\lim_{x \to \infty} f(x) = \infty\).

90. The degree is 5, and the leading coefficient is 5. Because the degree is odd and the leading coefficient is positive, \(\lim_{x \to -\infty} f(x) = -\infty\) and \(\lim_{x \to \infty} f(x) = \infty\).

91. It appears that \(f(x)\) has a relative minimum of \(-3\) at \(x = 0\) and a relative maximum of \(3\) at \(x = -2\).

92. It appears that \(f(x)\) has a relative maximum of \(8\) at \(x = 1\) and a relative minimum of \(-16\) at \(x = 3\) and \(x = -1\).

93. It appears that \(f(x)\) has a relative maximum of \(0\) at \(x = 0\) and \(x = -4\) and a relative minimum of \(-80\) at \(x = -2\) and \(-150\) at \(x = 2\).

### Differentiated Instruction

**Extension** Ask students if it is possible for a polynomial function to have a real zero greater than its greatest possible rational root. If so, ask students to give an example. Yes; sample answer: The function \(f(x) = 2x^2 - 5x + 1\) has 1 as its greatest possible rational zero, while its actual positive zero is \(\frac{5 + \sqrt{17}}{4}\) or about 2.28.
Rational Functions

A rational function \( f(x) \) is the quotient of two polynomial functions \( a(x) \) and \( b(x) \), where \( b(x) \neq 0 \).

\[
f(x) = \frac{a(x)}{b(x)}
\]

The domain of a rational function is the set of all real numbers excluding those values for which \( b(x) = 0 \) or the zeros of \( b(x) \).

One of the simplest rational functions is the reciprocal function \( f(x) = \frac{1}{x} \). The graph of the reciprocal function, like many rational functions, has branches that approach specific \( x \) - and \( y \) -values. The lines representing these values are called asymptotes.

The reciprocal function is undefined when \( x = 0 \), so its domain is \((-\infty, 0) \) or \((0, \infty) \). The behavior of \( f(x) = \frac{1}{x} \) to the left (\( 0^- \)) and right (\( 0^+ \)) of \( x = 0 \) can be described using limits.

\[
\lim_{x \to 0^-} f(x) = -\infty \quad \lim_{x \to 0^+} f(x) = \infty
\]

From Lesson 1-3, you should recognize \( 0 \) as a point of infinite discontinuity in the domain of \( f \). The line \( x = 0 \) in Figure 2.5.1 is called a vertical asymptote of the graph of \( f \). The end behavior of \( f \) can be also be described using limits.

\[
\lim_{x \to -\infty} f(x) = 0 \quad \lim_{x \to +\infty} f(x) = 0
\]

The line \( y = 0 \) in Figure 2.5.2 is called a horizontal asymptote of the graph of \( f \).

These definitions of vertical and horizontal asymptotes can be generalized.
You can use your knowledge of limits, discontinuity, and end behavior to determine the vertical and horizontal asymptotes, if any, of a rational function.

**Reading Math**

Limit Notation The expression \( \lim_{x \to c} f(x) \) is read as the limit of \( f \) of \( x \) as \( x \) approaches \( c \) from the left and the expression \( \lim_{x \to c^+} f(x) \) is read as the limit of \( f \) of \( x \) as \( x \) approaches \( c \) from the right.

**Key Concept: Vertical and Horizontal Asymptotes**

**Words**

- The line \( x = c \) is a *vertical asymptote* of the graph of \( f \) if \( \lim_{x \to c^-} f(x) = \pm \infty \) or \( \lim_{x \to c^+} f(x) = \pm \infty \).
- The line \( y = c \) is a *horizontal asymptote* of the graph of \( f \) if \( \lim_{x \to \pm \infty} f(x) = c \) or \( \lim_{x \to \pm \infty} f(x) = \pm \infty \).

**Example 1** Find Vertical and Horizontal Asymptotes

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any.

**a.** \( f(x) = \frac{x + 4}{x - 3} \)

**Step 1** Find the domain.

The function is undefined at the real zero of the denominator \( b(x) = x - 3 \). The real zero of \( b(x) = 3 \). Therefore, the domain of \( f \) is all real numbers except \( x = 3 \).

**Step 2** Find the asymptotes, if any.

**Check for vertical asymptotes.**

Determine whether \( x = 3 \) is a point of infinite discontinuity. Find the limit as \( x \) approaches \( 3 \) from the left and the right.

<table>
<thead>
<tr>
<th>( x )</th>
<th>2.9</th>
<th>2.99</th>
<th>3</th>
<th>3.01</th>
<th>3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x))</td>
<td>-68</td>
<td>-69</td>
<td>undefined</td>
<td>7001</td>
<td>701</td>
</tr>
</tbody>
</table>

Because \( \lim_{x \to 3^-} f(x) = -\infty \) and \( \lim_{x \to 3^+} f(x) = \infty \), you know that \( x = 3 \) is a vertical asymptote of \( f \).

**Check for horizontal asymptotes.**

Use a table to examine the end behavior of \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-10,000</th>
<th>-1000</th>
<th>-100</th>
<th>0</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x))</td>
<td>0.9993</td>
<td>0.9930</td>
<td>0.9320</td>
<td>-1.33</td>
<td>1.0722</td>
<td>1.0070</td>
<td>1.0007</td>
</tr>
</tbody>
</table>

The table suggests that \( \lim_{x \to \pm \infty} f(x) = 1 \). Therefore, you know that \( y = 1 \) is a horizontal asymptote of \( f \).

**CHECK**

The graph of \( f(x) = \frac{x + 4}{x - 3} \) shown supports each of these findings.✓

**Focus on Mathematical Content**

**Asymptotes** Asymptotes can be horizontal, vertical, or slanted lines. Asymptotes can be determined by observing the limits, discontinuity, and end behavior of the rational function.

**Rational Functions**

Examples 1–4 show how to analyze and graph the rational function \( f(x) = \frac{a(x)}{b(x)} \) by analyzing the real zeros of \( b(x) \) to find the domain and by using limits or by comparing the degree of \( a(x) \) to the degree of \( b(x) \) to find the equations of the vertical, horizontal, and oblique asymptotes. **Example 5** shows how there can be removable discontinuities in a rational function.

**Formative Assessment**

Use the Guided Practice exercises after each example to determine students’ understanding of concepts.
1. Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any.
   a. \( f(x) = \frac{x}{x-1} \)
      \( D = \{x | x \neq 1, x \in \mathbb{R}\} \); vertical asymptote at \( x = 1 \); horizontal asymptote at \( y = 1 \)
   b. \( f(x) = 4x^2 + \frac{3}{2x^2 + 1} \)
      \( D = \{x | x \in \mathbb{R}\} \); no vertical asymptotes; horizontal asymptote at \( y = 2 \)

**Additional Examples** also in Interactive Classroom PowerPoint® Presentations

**Teach with Tech**

**Student Response System** Show students the equation of a rational function. Ask them if the function has a vertical asymptote. Have students respond A for yes and B for no.

**Tips for New Teachers**

**Rational Functions** The relationships described in the Key Concept box assume that the rational function \( f \) cannot be reduced to a constant function.

**KeyConcept** Graphs of Rational Functions

If \( f \) is the rational function given by

\[
  f(x) = \frac{a(x)}{b(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}
\]

where \( b(x) \neq 0 \) and \( a(x) \) and \( b(x) \) have no common factors other than \( \pm 1 \), then the graph of \( f \) has the following characteristics.

**Vertical Asymptotes** Vertical asymptotes may occur at the real zeros of \( b(x) \).

**Horizontal Asymptote** The graph has either one or no horizontal asymptotes as determined by comparing the degree \( n \) of \( a(x) \) to the degree \( m \) of \( b(x) \).

- If \( n < m \), the horizontal asymptote is \( y = 0 \).
- If \( n = m \), the horizontal asymptote is \( y = \frac{a_n}{b_m} \).
- If \( n > m \), there is no horizontal asymptote.

**Intercepts** The \( x \)-intercepts, if any, occur at the real zeros of \( a(x) \). The \( y \)-intercept, if it exists, is the value of \( f \) when \( x = 0 \).

**Differentiated Instruction**

**Extension** Use after Example 1

Ask students to determine the horizontal asymptotes of

\[
  f(x) = \frac{x}{x - 4}, \quad f(x) = \frac{x^2}{x^2 - 4}, \quad \text{and} \quad f(x) = \frac{x^3}{x^3 - 4}
\]

Then have students make a conjecture about the asymptotes for \( f(x) = \frac{x^n}{x^n - 4} \) for any positive integer \( n \). Each function has \( y = 1 \) as a horizontal asymptote.
To graph a rational function, simplify $f$, if possible, and then follow these steps.

**Step 1.** Find the domain.

**Step 2.** Find and sketch the asymptotes, if any.

**Step 3.** Find and plot the $x$-intercepts and $y$-intercept, if any.

**Step 4.** Find and plot at least one point in the test intervals determined by any $x$-intercepts and vertical asymptotes.

### Example 2: Graph Rational Functions: $n < m$ and $n > m$

For each function, determine any vertical and horizontal asymptotes and intercepts. Then graph the function, and state its domain.

#### a. $g(x) = \frac{6}{x + 3}$

**Step 1.** The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq 3, x \in \mathbb{R}\}$.

**Step 2.** There is a vertical asymptote at $x = -3$.

**Step 3.** The polynomial in the numerator has no real zeros, so $g$ has no $x$-intercepts. Because $g(0) = 2$, the $y$-intercept is 2.

**Step 4.** Graph the asymptotes and intercepts. Then choose $x$-values that fall in the test intervals determined by the vertical asymptote to find additional points to plot on the graph. Use smooth curves to complete the graph.

#### b. $k(x) = \frac{x^2 - 7x + 10}{x - 3}$

Factoring the numerator yields $k(x) = \frac{(x - 2)(x - 5)}{x - 3}$. Notice that the numerator and denominator have no common factors, so the expression is in simplest form.

**Step 1.** The function is undefined at $b(x) = 0$, so the domain is $\{x \mid x \neq 3, x \in \mathbb{R}\}$.

**Step 2.** There is a vertical asymptote at $x = 3$.

**Step 3.** The numerator has zeros at $x = 2$ and $x = 5$, so the $x$-intercepts are 2 and 5. $k(0) = -\frac{10}{3}$, so the $y$-intercept is at about $-3.3$.

**Step 4.** Graph the asymptotes and intercepts. Then find and plot points in the test intervals determined by the intercepts and vertical asymptotes: $(-\infty, 0), (0, 3), (3, \infty)$. Use smooth curves to complete the graph.

---

### Additional Example

2. For each function, determine any vertical and horizontal asymptotes and intercepts. Then graph the function and state its domain.

a. $k(x) = \frac{7}{x + 5}$

vertical asymptote at $x = -5$; horizontal asymptote at $y = 0$; $y$-intercept: 1.4

$$D = \{x \mid x \neq -5, x \in \mathbb{R}\}$$

b. $f(x) = \frac{x + 1}{x^2 - 4}$

vertical asymptotes at $x = 2$ and $x = -2$; horizontal asymptote at $y = 0$; $x$-intercept: $-1$; $y$-intercept: $-0.25$;

$$D = \{x \mid x \neq 2, -2, x \in \mathbb{R}\}$$

---

### Guided Practice

2A. $h(x) = \frac{2}{x^2 + 2x - 3}$

2A. See margin.

2B. $m(x) = \frac{x}{x^2 + x - 2}$

---

2B. vertical asymptotes at $x = -2$ and $x = 1$; horizontal asymptote at $y = 0$; $x$-intercept: 0; $y$-intercept: 0; $D = \{x \mid x \neq -2, 1, x \in \mathbb{R}\}$
Example 3 Graph a Rational Function: \( n = m \)

Determine any vertical and horizontal asymptotes and intercepts for \( f(x) = \frac{3x^2 - 3}{x^2 - 9} \).

Then graph the function, and state its domain.

Factoring both numerator and denominator yields \( f(x) = \frac{3(x - 1)(x + 1)}{(x - 3)(x + 3)} \) with no common factors.

**Step 1**
The function is undefined at \( b(x) = 0 \), so the domain is \( \{ x \mid x \neq -3, 3, x \in \mathbb{R} \} \).

**Step 2**
There are vertical asymptotes at \( x = 3 \) and \( x = -3 \).

There is a horizontal asymptote at \( y = \frac{3}{3} \) or \( y = 1 \), the leading coefficients of the numerator and denominator, because the degrees of the polynomials are equal.

**Step 3**
The \( x \)-intercepts are 1 and \(-1\), the zeros of the numerator. The \( y \)-intercept is \( \frac{1}{3} \) because \( f(0) = \frac{1}{3} \).

**Step 4**
Graph the asymptotes and intercepts. Then find and plot points in the test intervals \( (-\infty, -3), (-3, -1), (-1, 1), (1, 3), \) and \( (3, \infty) \).

When the degree of the numerator is exactly one more than the degree of the denominator, the graph has a slant or oblique asymptote.

**Key Concept** Oblique Asymptotes

If \( f \) is the rational function given by

\[
f(x) = \frac{a(x)}{b(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}
\]

where \( b(x) \) has a degree greater than 0 and \( a(x) \) and \( b(x) \) have no common factors other than 1, then the graph of \( f \) has an oblique asymptote if \( n = m + 1 \). The function for the oblique asymptote is the quotient polynomial \( q(x) \) resulting from the division of \( a(x) \) by \( b(x) \).

\[
f(x) = \frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}
\]

function for oblique asymptote
**Example 4** Graph a Rational Function: \( n = m + 1 \)

Determine any asymptotes and intercepts for \( f(x) = \frac{2x^3}{x^2 + x - 12} \). Then graph the function, and state its domain.

Factoring the denominator yields \( f(x) = \frac{2x^3}{(x + 4)(x - 3)} \).

**Step 1** The function is undefined at \( x = -4 \) and \( x = 3 \). Between the vertical asymptotes at \( x = -4 \) and \( x = 3 \), however, the graph crosses the line \( y = 2x - 2 \). For this reason, a slant or horizontal asymptote is sometimes referred to as an end-behavior asymptote.

**Step 2** There are vertical asymptotes at \( x = -4 \) and \( x = 3 \).

The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote. Because the degree of the numerator is exactly one more than the degree of the denominator, \( f \) has a slant asymptote. Using polynomial division, you can write the following:

\[
f(x) = \frac{2x^3}{x^2 + x - 12} = -2x - 2 + \frac{26x - 24}{x^2 + x - 12}
\]

Therefore, the equation of the slant asymptote is \( y = 2x - 2 \).

**Step 3** The \( x \)- and \( y \)-intercepts are 0 because 0 is the zero of the numerator and \( f(0) = 0 \).

**Step 4** Graph the asymptotes and intercepts. Then find and plot points in the test intervals \((-\infty, -4), (-4, 0), (0, 3), \) and \((3, \infty)\).

When the numerator and denominator of a rational function have common factors, the graph of the function has removable discontinuities called holes, at the zeros of the common factors. Be sure to indicate these points of discontinuity when you graph the function.

\[
f(x) = \frac{(x - b)(x - c)}{(x - b)(x - c)}
\]

Divide out the common factor in the numerator and denominator. The zero of \( x = a \) is a.

---

**Additional Example**

Determine any asymptotes and intercepts for \( f(x) = \frac{x^2 + x - 8}{x + 3} \).

Then graph the function, and state its domain. **Vertical asymptote at** \( x = -3 \); **slant asymptote at** \( y = x - 2 \);

\[
x\text{-intercepts: } \frac{-1 + \sqrt{33}}{2}, \frac{-1 - \sqrt{33}}{2} ; \ y\text{-intercept: } \frac{-8}{3}
\]

**Additional Answers**

**Guided Practice**

For each function, determine any asymptotes and intercepts. Then graph the function and state its domain. 4A–B. See margin.

4A. \( h(x) = \frac{x^2 + 3x - 3}{x + 4} \)

4B. \( p(x) = \frac{x^2 - 4x + 1}{2x - 3} \)

---

4B. **vertical asymptote at** \( x = \frac{3}{2} \); **oblique asymptote at** \( y = \frac{1}{2}x - \frac{5}{4} \)

**x-intercepts:** about 0.27, about 3.73; **y-intercept:** \(-\frac{1}{3}\)

\[
D = \left\{ x | x \neq \frac{3}{2}, x \in \mathbb{R} \right\}
\]
### Additional Example

5 Determine any vertical and horizontal asymptotes, holes, and intercepts for \( h(x) = \frac{x^2 - 9}{x^2 - x - 6} \).

Then graph the function and state its domain. Vertical asymptote at \( x = -2 \); horizontal asymptote at \( y = 1 \); x-intercept: \(-3\); y-intercept: \( \frac{3}{2} \); hole: \( (\frac{3}{2}, \frac{6}{5}) \).

**D** = \( \{ x | x \neq -2, 3, x \in \mathbb{R} \} \)

---

### Tips for New Teachers

**Holes in Graphs** When the numerator and denominator of a rational function have a common factor \( (x - c) \), the point \( (c, f(c)) \) needs to be omitted from the graph. It is indicated by a circle or hole. If \( x = c \) is a vertical asymptote, there is no hole in the graph.

### 2 Rational Equations

**Examples 6–8** show how to solve rational equations involving fractions. All terms are multiplied by the LCD and then the variable is isolated. Solutions should be checked in the original equation to identify any extraneous solutions.

### Additional Example

6 Solve \( x - \frac{4}{x - 6} = 0 \). \( 3 \pm \sqrt{13} \)

---

### Example 5 Graph a Rational Function with Common Factors

Determine any vertical and horizontal asymptotes, holes, and intercepts for \( h(x) = \frac{x^2 - 4}{x^2 - 2x - 8} \).

Then graph the function and state its domain.

Factoring both the numerator and denominator yields \( h(x) = \frac{(x - 2)(x + 2)}{(x - 4)(x + 2)} \) or \( x = 4 \), \( x \neq -2 \).

**Step 1** The function is undefined at \( h(0) = 0 \), so the domain is \( \{ x | x \neq -2, 4, x \in \mathbb{R} \} \).

**Step 2** There is a vertical asymptote at \( x = 4 \), the real zero of the simplified denominator.

**Step 3** There is a horizontal asymptote at \( y = \frac{1}{2} \) or \( 1 \), the ratio of the leading coefficients of the numerator and denominator, because the degrees of the polynomials are equal.

**Step 4** The x-intercept is \( 2 \), the zero of the simplified numerator. The y-intercept is \( \frac{1}{2} \) because \( h(0) = \frac{1}{2} \).

**Step 5** Graph the asymptotes and intercepts. Then find and plot points in the test intervals \( (-\infty, 2), (2, 4), \) and \( (4, \infty) \).

There is a hole at \( \left( -2, \frac{2}{3} \right) \) because the original function is undefined when \( x = -2 \).

---

### Example 6 Solve a Rational Equation

Solve \( x + \frac{6}{x - 8} = 0 \).

Original equation

\[ x + \frac{6}{x - 8} = 0 \]

Multiply by the LCD, \( x - 8 \).

\[ x(x - 8) + \frac{6}{x - 8} = 0 \]

Distributive Property

\[ x^2 - 8x + 6 = 0 \]

Quadratic Formula

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(6)}}{2(1)} \]

Simplify.

\[ x = 4 \pm \sqrt{10} \]

or \( 2 \pm \sqrt{10} \)

---

### Guided Practice

For each function, determine any vertical and horizontal asymptotes, holes, and intercepts. Then graph the function and state its domain.

5A. \( g(x) = \frac{x^2 + 10x + 24}{x^2 + x - 12} \) See margin.

5B. \( c(x) = \frac{x^2 - 2x - 3}{x^2 - 4x - 5} \) See margin.

---

### Differentiated Instruction

**Verbal/Linguistic Learners** Divide students into groups of three or four. Write a rational equation on the board and have each group solve it, writing the steps they use to find the solution. Then have groups compare and contrast the processes they used.
Solving a rational equation can produce extraneous solutions. Always check your answers in the original equation.

**Example 7  Solve a Rational Equation with Extraneous Solutions**

Solve \( \frac{4}{x^2 - 6x + 8} = \frac{3x}{x - 2} + \frac{2}{x - 4} \)

The LCD of the expressions is \((x - 2)(x - 4)\), which are the factors of \(x^2 - 6x + 8\).

\[
\frac{4}{x^2 - 6x + 8} = \frac{3x}{x - 2} + \frac{2}{x - 4}
\]

Original equation

\[
\frac{4}{x^2 - 6x + 8} = \frac{(x - 2)(x - 4)}{x^2 - 6x + 8} \cdot \left( \frac{3x}{x - 2} + \frac{2}{x - 4} \right)
\]

Multiply by the LCD.

\[
4 = 3x(x - 4) + 2(x - 2)
\]

Distributive Property

\[
4 = 3x^2 - 10x - 4
\]

Distributive Property

\[
0 = 3x^2 - 10x - 4
\]

Subtract 4 from each side.

\[
0 = (3x + 2)(x - 2)
\]

Factor.

\[
x = -\frac{2}{3}
\]

or \(x = 4\)

Solve.

Because the original equation is not defined when \(x = 4\), you can eliminate this extraneous solution. So, the only solution is \(-\frac{2}{3}\).

**Guided Practice**

Solve each equation.

7A. \( \frac{2x}{x + 3} + \frac{3}{x - 6} = \frac{27}{x^2 - 3x - 18} \)

7B. \( \frac{12}{x^2 + 6x} = \frac{2}{x + 6} + \frac{x - 2}{x} \)

**Real-World Example 8  Solve a Rational Equation**

**ELECTRICITY** The diagram of an electric circuit shows three parallel resistors. If \(R\) is the equivalent resistance of the three resistors, then \(\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\). In this circuit, \(R_1\) is twice the resistance of \(R_2\), and \(R_1\) equals 20 ohms. Suppose the equivalent resistance is equal to 10 ohms. Find \(R_2\) and \(R_3\).

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}
\]

Original equation

\[
\frac{1}{10} = \frac{1}{2R_2} + \frac{1}{R_2} + \frac{1}{20}
\]

Subtract \(\frac{1}{20}\) from each side.

\[
\frac{1}{20} = \frac{1}{2R_2} + \frac{1}{R_2}
\]

Multiply each side by the LCD, \(20R_2^2\).

\[
\frac{20R_2}{20} = \frac{20R_2}{20} \cdot \left( \frac{1}{2R_2} + \frac{1}{R_2} \right)
\]

Simplify.

\[
R_2 = 10 + 20 \text{ or } 30
\]

\(R_2 = 30\) ohms and \(R_1 = 2R_3\) or 60 ohms.

**Guided Practice**

8. **ELECTRONICS** Suppose the current \(I\), in amps, in an electric circuit is given by the formula \(I = t + \frac{1}{10 - t}\), where \(t\) is time in seconds. At what time is the current 1 amp?

**Additional Examples**

7 Solve \(x + \frac{x}{x - 1} = \frac{3x - 2}{x - 1}\).

8 **WATER CURRENT** The rate of the water current in a river is 4 miles per hour. In 2 hours, a boat travels 6 miles with the current to one end of the river and 6 miles back. If \(r\) is the rate of the boat in still water, \(r + 4\) is its rate with the current, \(r - 4\) is its rate against the current, and \(\frac{6}{r + 4} + \frac{6}{r - 4} = 2\), find \(r\).

**Additional Answers (Guided Practice)**

5A. vertical asymptote at \(x = 3\);
horizontal asymptote at \(y = 1\);
x-intercept: -6; y-intercept: -2;
hole: \((-4, -\frac{2}{7})\)

\[D = \{x \mid x \neq -4, 3, x \in \mathbb{R}\}\]

5B. vertical asymptote at \(x = 5\);
horizontal asymptote at \(y = 1\);
x-intercept: 3; y-intercept: \(\frac{3}{5}\);
hole: \((-1, -\frac{2}{3})\); \(D = \{x \mid x \neq -1, 5, x \in \mathbb{R}\}\)
Exercises

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any. (Example 1) 1–8. See margin.

1. \( f(x) = \frac{x^2 - 2}{x^2 - 4} \) 2. \( h(x) = \frac{2}{x + 4} \)

3. \( f(x) = \frac{x(x - 1)(x + 2)^2}{(x + 3)(x - 4)} \) 4. \( g(x) = \frac{x - 6}{(x + 3)(x + 5)} \)

5. \( h(x) = \frac{x^2 - 4x + 1}{x^2 + 2x} \) 6. \( f(x) = \frac{2x^4 + 9x + 20}{x - 4} \)

7. \( h(x) = \frac{x^2 - 2x + 1}{x^2 - 2x(x + 1)} \) 8. \( g(x) = \frac{(x - 4)(x + 2)}{(x + 1)(x - 3)} \)


For each function, determine any asymptotes and intercepts. Then graph the function, and state its domain. (Examples 2–6)

9. \( f(x) = \frac{x + 2(x - 3)}{x + 4(x - 4)} \) 10. \( g(x) = \frac{(2x + 3)(x - 6)}{(x + 3)(x - 1)} \)

11. \( h(x) = \frac{8}{(x - 2)(x + 2)} \) 12. \( f(x) = \frac{x + 2}{3x - 6} \)

13. \( g(x) = \frac{(x + 2)(x + 5)}{(x + 5)(x - 6)} \) 14. \( h(x) = \frac{(x + 6)(x + 4)}{3(x - 5)(x + 2)} \)

15. \( b(x) = \frac{x^2(x - 2)(x + 5)}{x^2 + 4x + 3} \) 16. \( f(x) = \frac{x^2 + 6x}{x^2 - 5x - 24} \)

17. \( h(x) = \frac{x^2 - 5x - 6}{x^2 + 4x + 5} \) 18. \( g(x) = \frac{4}{x^2 + 6} \)

19. \( \text{SALES} \) The business plan for a new car wash projects that profits in thousands of dollars will be modeled by the function \( p(z) = \frac{3z^2 - 3}{2z^2 + 7z + 5} \), where \( z \) is the week of operation and \( z = 0 \) represents opening. (Example 4)

a. State the domain of the function.

b. Determine any vertical and horizontal asymptotes and intercepts for \( p(z) \).

c. Graph the function. See Chapter 2 Answer Appendix.

For each function, determine any asymptotes, holes, and intercepts. Then graph the function and state its domain. (Examples 2–5) 20–29. See Chapter 2 Answer Appendix.

20. \( h(x) = \frac{3x^2 - 4}{x^2} \) 21. \( h(x) = \frac{4x^4 - 2x + 1}{3x^3 - 4} \)

22. \( f(x) = \frac{x^2 + 2x - 15}{x^2 + 4x + 3} \) 23. \( g(x) = \frac{x + 7}{x - 4} \)

24. \( h(x) = \frac{x^2}{3x} \) 25. \( g(x) = \frac{x^3 + 3x^2 + 2x}{x - 4} \)

26. \( f(x) = \frac{x^2 - 4x - 21}{x^2 + 2x - 5x - 6} \) 27. \( g(x) = \frac{x^2 - 4}{x^2 + 3x^2 - 4x - 4} \)

28. \( f(x) = \frac{x + 4(x - 1)}{(x - 1)(x - 3)} \) 29. \( g(x) = \frac{(2x + 1)(x - 5)}{(x - 5)(x + 4)} \)

30. \( \text{STATISTICS} \) A number \( x \) is said to be the harmonic mean of \( y \) and \( z \) if \( \frac{1}{x} \) is the average of \( \frac{1}{y} \) and \( \frac{1}{z} \). (Example 7)

a. Write an equation for which the solution is the harmonic mean of 30 and 45.

b. Find the harmonic mean of 30 and 45.

31. \( \text{OPTICS} \) The lens equation is \( \frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2} \), where \( f \) is the focal length, \( d_1 \) is the distance from the lens to the image, and \( d_2 \) is the distance from the lens to the object. Suppose the object is 32 centimeters from the lens and the focal length is 8 centimeters. (Example 7)

32. Solve each equation. (Examples 6–8) 35. No solution

33. \( \frac{x - 1}{2x} + \frac{x + 2}{3x} = 1 \) 34. \( \frac{2}{y} + \frac{3}{2} \) 35. \( y^2 = 1 \) 36. \( y^2 = 4 \)

37. \( a = \frac{2}{x - 1} \) 38. \( x - 1 = 4 \) 39. \( x = -3 \) 40. \( x = 2 \)

41. \( x = 2 \) 42. WATER The cost per day to remove \( p \) percent of the salt from seawater at a desalination plant is \( c(x) = \frac{94.5}{100 - x} \), where \( 0 \leq x < 100 \).

a. Graph the function using a graphing calculator.

b. Graph the line \( y = 8000 \) and find the intersection with the graph of \( c(x) \) to determine what percent of salt can be removed for \$8000 per day.

c. According to the model, is it feasible for the plant to remove 100% of the salt? Explain your reasoning.

a–c. See Chapter 2 Answer Appendix.

Write a rational function for each set of characteristics.

43. \( x \)-intercepts at \( x = 0 \) and \( x = 4 \), vertical asymptotes at \( x = 1 \) and \( x = 6 \), and a horizontal asymptote at \( y = 0 \)

44–44. See margin.

44. \( x \)-intercepts at \( x = 2 \) and \( x = -3 \), vertical asymptote at \( x = 4 \), and point discontinuity at \( x = -5 \).

3 Practice

Additional Answers

1. \( D = \{ x \mid x \neq 2, -2, x \in \mathbb{R} \}; x = 2, x = -2, y = 1 \)

2. \( D = \{ x \mid x \neq -4, x \in \mathbb{R} \}; x = -4 \)

3. \( D = \{ x \mid x \neq -4, -3, x \in \mathbb{R} \}; x = 4, x = -3 \)

4. \( D = \{ x \mid x \neq -3, -5, x \in \mathbb{R} \}; x = -3, x = -5, y = 0 \)

5. \( D = \{ x \mid x \neq 0, -2, x \in \mathbb{R} \}; x = 0, x = -2, y = 2 \)

6. \( D = \{ x \mid x \neq 4, x \in \mathbb{R} \}; x = 4 \)

7. \( D = \{ x \mid x \neq -2, -4, x \in \mathbb{R} \}; x = 2, x = -4, y = 0 \)

8. \( D = \{ x \mid x \neq 3, -1, x \in \mathbb{R} \}; x = 3, x = -1, y = 1 \)

Differentiated Homework Options

<table>
<thead>
<tr>
<th>Level</th>
<th>Assignment</th>
<th>Two-Day Option</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AL</strong> Approaching Level</td>
<td>1–42, 54, 55, 57, 59–76</td>
<td>1–41 odd 2–42 even, 54, 55, 57, 59–76</td>
</tr>
<tr>
<td><strong>BL</strong> Beyond Level</td>
<td>43–76</td>
<td></td>
</tr>
</tbody>
</table>
53. **Multiple Representations** In this problem, you will investigate asymptotes of rational functions.

### a. Tabular

<table>
<thead>
<tr>
<th>Function</th>
<th>Horizontal Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{x^2 - 5x + 4}{x^2 + 2} )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>( h(x) = \frac{x^3 - 4x^2 + 4x - 4}{x^2 - 4} )</td>
<td>( y = 0 )</td>
</tr>
<tr>
<td>( g(x) = \frac{x^2 - 1}{x^2 + 3} )</td>
<td>( y = 0 )</td>
</tr>
</tbody>
</table>

### b. Graphical

Graph each function and its horizontal asymptote from part a.

### c. Tabular

<table>
<thead>
<tr>
<th>Function</th>
<th>Real Zeros of Numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{x^2 - 5x + 4}{x^2 + 2} )</td>
<td>1, 4</td>
</tr>
<tr>
<td>( h(x) = \frac{x^3 - 4x^2 + 4x - 4}{x^2 - 4} )</td>
<td>3</td>
</tr>
<tr>
<td>( g(x) = \frac{x^2 - 1}{x^2 + 3} )</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

### d. Verbal

Make a conjecture about the behavior of the graph of a rational function when the degree of the numerator is greater than the degree of the denominator and the numerator has at least one real zero.

#### H.O.T. Problems

**Use Higher-Order Thinking Skills**

54. **Reasoning** Given \( f(x) = \frac{x^2 + 2x + 2}{x^2 + 1} \), will \( f(x) \) sometimes, always, or never have a horizontal asymptote at \( y = 1 \) if \( a, b, c, d, e \), and \( f \) are constants with \( a \neq 0 \) and \( d \neq 0 \). Explain.

**See Chapter 2 Appendix.**

55. **PREWRITE** Design a lesson plan to teach the graphing rational functions topics in this lesson. Make a plan that addresses purpose, audience, a controlling idea, logical sequence, and time frame for completion.

**See students' work.**

56. **CHALLENGE** Write a rational function that has vertical asymptotes at \( x = -2 \) and \( x = 3 \) and an oblique asymptote \( y = 3x \). **Sample answer:** \( f(x) = \frac{3x^3 - 3x^2}{x - 4} + 6 \)

57. **Writing in Math** Use words, graphs, tables, and equations to show how to graph a rational function.

**See students' work.**

58. **Challenge** Solve for \( k \) so that the rational equation has exactly one extraneous solution and one real solution.

59. **Writing in Math** Explain why all of the test intervals must be used in order to get an accurate graph of a rational function. **See Chapter 2 Appendix Answer.**

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**Additional Answers**

69. \[ f(x) = (x + 7)^2 \]

70. \[ f(x) = (x - 4)^2 \]

71. \[ f(x) = x^4 - 5 \]

73. SAT/ACT A company sells ground coffee in two sizes of cylindrical containers. The smaller container holds 10 ounces of coffee. If the larger container has twice the radius of the smaller container and 1.5 times the height, how many ounces of coffee does the larger container hold? (The volume of a cylinder is given by the formula \( V = \pi r^2 h \).) C

- A 30
- B 45
- C 60
- D 75
- E 90

74. What are the solutions of \( 1 = \frac{x}{3} + \frac{2}{3} \)? H

- F \( x = 1, x = -2 \)
- G \( x = -2, x = 1 \)
- H \( x = 1 + \sqrt{3}, x = 1 - \sqrt{3} \)
- J \( x = \frac{1 + \sqrt{3}}{2}, x = \frac{1 - \sqrt{3}}{2} \)

75. REVIEW Alex wanted to determine the average of his 6 test scores. He added the scores correctly to get \( T \) but divided by 7 instead of 6. The result was 12 less than his actual average. Which equation could be used to determine the value of \( T \)? C

- A \( 67 + 12 = 7T \)
- B \( \frac{T}{6} = \frac{T - 12}{6} \)
- C \( \frac{T}{7} + 12 = \frac{T}{6} \)
- D \( \frac{T}{2} = \frac{T - 12}{2} \)

76. Diana can put a puzzle together in three hours. Ella can put the same puzzle together in five hours. How long will it take them if they work together? J

- F \( 1\frac{2}{3} \) hours
- G \( 1\frac{3}{5} \) hours
- H \( 1\frac{3}{7} \) hours

**Spiral Review**

List all the possible rational zeros of each function. Then determine which, if any, are zeros. (Lesson 2-4)

60. \( f(x) = x^3 + 2x^2 - 5x - 6 \)
61. \( f(x) = x^3 - 2x^2 + x + 18 \)
62. \( f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 \)

\[ \pm 1, \pm 2, \pm 3, \pm 6, -3, -1, 2 \]

\[ \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18; -2 \]

63. \( f(x) = x^4 + 3x^2 - 2x + 1 \)
64. \( f(x) = 2x^4 - 5x^3 - 11x^2 - 4x - 2 \)
65. \( f(x) = 4x^2 - 9 \)
66. \( f(x) = (x - 2)(x^3 - 13x - 12) \)

\[ (x - 2)(x + 1)(x + 3)^2 \]

67. \( f(x) = 2x^4 - 5x^3 + 10x^2 - 2x - 1 \)
68. \( f(x) = 4x^4 - 5x^3 + 10x^2 - 2x - 1 \)
69. \( f(x) = (x - 2)(x + 1)(x + 1)(x - 1) \)
70. \( f(x) = (x - 2)(x + 1)(x + 1)(x - 1) \)
71. \( f(x) = x^4 - 5 \)

**Skills Review for Standardized Tests**

72. RETAIL Sara is shopping at a store that offers $10 cash back for every $50 spent. Let \( f(x) = \frac{x}{50} \) and \( h(x) = 10x \), where \( x \) is the amount of money Sara spends. (Lesson 1-6)

a. If Sara spends money at the store, is the cash back bonus represented by \( h(f(x)) \) or \( f(h(x)) \)? Explain your reasoning.

b. Determine the cash back bonus if Sara spends $312.68 at the store. $60

Follow-up

Students have explored modeling using power, radical, polynomial, and rational functions.

**Ask:**

- What are the limitations of mathematical modeling? Sample answer: Not all real-world situations can be modeled. For those that can be modeled, predictions that are made using the model may not be accurate when based on data values that are outside the range of data values used to create the model. Therefore, after a model is created, it should be carefully analyzed before it is used to make predictions/decisions.

140 | Lesson 2-5 | Rational Functions
1 Polynomial Inequalities If \( f(x) \) is a polynomial function, then a polynomial inequality has the general form \( f(x) \leq 0 \), \( f(x) < 0 \), \( f(x) 
eq 0 \), \( f(x) > 0 \), or \( f(x) \geq 0 \). The inequality \( f(x) < 0 \) is true when \( f(x) \) is negative, while \( f(x) > 0 \) is true when \( f(x) \) is positive.

In Lesson 1-2, you learned that the \( x \)-intercepts of a polynomial function are the real zeros of the function. When ordered, these zeros divide the \( x \)-axis into intervals for which the value of \( f(x) \) is either entirely positive (above the \( x \)-axis) or entirely negative (below the \( x \)-axis).

By finding the sign of \( f(x) \) for just one \( x \)-value in each interval, you can determine on which intervals the function is positive or negative. From the test intervals represented by the sign chart at the right, you know that:

- \( f(x) < 0 \) on \((-4, -2) \cup (2, 5) \cup (5, \infty)\),
- \( f(x) \leq 0 \) on \([-4, -2] \cup [2, \infty)\),
- \( f(x) = 0 \) at \( x = -4, 2, 5 \),
- \( f(x) > 0 \) on \((-\infty, -4) \cup (-2, 2)\), and
- \( f(x) \geq 0 \) on \((-\infty, -4] \cup [-2, 2] \cup [5, \infty)\).

Example 1 Solve a Polynomial Inequality

Solve \( x^2 - 6x - 30 > -3 \).

Adding 3 to each side, you get \( x^2 - 6x - 27 > 0 \). Let \( f(x) = x^2 - 6x - 27 \). Factoring yields \( f(x) = (x + 3)(x - 9) \), so \( f(x) \) has real zeros at \(-3 \) and \(9\). Create a sign chart using these zeros.

Then substitute an \( x \)-value in each interval to determine if \( f(x) \) is positive or negative at that point.

Because \( f(x) \) is positive on the first and last intervals, the solution set of \( x^2 - 6x - 30 > -3 \) is \((-\infty, -3) \cup (9, \infty)\). The graph of \( f(x) \) supports this conclusion, because \( f(x) \) is above the \( x \)-axis on these same intervals.

Guided Practice

Solve each inequality.

1A. \( x^2 + 5x + 6 < 20 \) \((-7, 2)\)
1B. \( (x - 4)^2 > 4 \) \((-\infty, 2) \cup (6, \infty)\)

Lesson 2-6 Resources

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<th>On Level OL</th>
<th>Beyond Level EL</th>
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<td>Differentiated Instruction</td>
<td>Differentiated Instruction</td>
<td>Differentiated Instruction</td>
<td>Study Guide and Intervention Practice Word Problem Practice Graphic Calculator Activity</td>
</tr>
<tr>
<td>Other</td>
<td>Study Notebook 5-Minute Check</td>
<td>Study Notebook 5-Minute Check</td>
<td>Study Notebook 5-Minute Check</td>
<td>Study Notebook 5-Minute Check</td>
</tr>
</tbody>
</table>
Based on your answers, what conclusion can you make about the solutions of $2x - 6 > 0$? $x > 3$

Why do you think 3 is a critical solution. When you know what value makes the equation true, you can test intervals on either side of that value to find what makes the inequality true. This inequality is positive for values greater than 3, so $x > 3$ is the solution.

### 1 Polynomial Inequalities

Examples 1–3 show how to solve polynomial inequalities using the real zeros of the related polynomial function, their multiplicity, and the function’s end behavior along with a sign chart.

#### Formative Assessment

Use the Guided Practice exercises after each example to determine students’ understanding of concepts.

#### Additional Examples

1. Solve $x^2 - 8x + 16 \leq 1$. $[3, 5]$
2. Solve $x^3 - 22x > 3x^2 - 24$. $(-4, 1) \cup (6, \infty)$
3. Solve each inequality.
   a. $x^2 + 2x + 3 < 0$ Ø
   b. $x^2 + 2x + 3 \geq 0$ $(-\infty, \infty)$
   c. $x^2 + 12x + 36 > 0$ $(-\infty, -6) \cup (-6, \infty)$
   d. $x^2 + 12x + 36 \leq 0$ $[-6]$

#### Guided Practice

Solve each inequality.

2A. $2x^2 - 10x \leq 2x - 16$ $[2, 4]$
2B. $2x^3 + 7x^2 - 12x - 45 \geq 0$ $[-3, \infty)$

When a polynomial function does not intersect the $x$-axis, the related inequalities have unusual solutions.

#### Example 3 Polynomial Inequalities with Unusual Solution Sets

Solve each inequality.

a. $x^2 + 5x + 8 < 0$
   The related function $f(x) = x^2 + 5x + 8$ has no real zeros, so there are no sign changes. This function is positive for all real values of $x$. Therefore, $x^2 + 5x + 8 < 0$ has no solution. The graph of $f(x)$ supports this conclusion, because the graph is never on or below the $x$-axis. The solution set is Ø.

b. $x^2 + 5x + 8 \geq 0$
   Because the related function $f(x) = x^2 + 5x + 8$ is positive for all real values of $x$, the solution set of $x^2 + 5x + 8 \geq 0$ is all real numbers or $(-\infty, \infty)$. 

If you know the real zeros of a function, including their multiplicity, and the function’s end behavior, you can create a sign chart without testing values.

#### Example 2 Solve a Polynomial Inequality Using End Behavior

Solve $3x^3 = 4x^2 - 13x - 6 \leq 0$.

Step 1: Let $f(x) = 3x^3 - 4x^2 - 13x - 5$. Use the techniques from Lesson 2-4 to determine that $f$ has real zeros with multiplicity 1 at $-\frac{3}{5}$ and 3. Set up a sign chart.

Step 2: Determine the end behavior of $f(x)$.
Because the degree of $f$ is odd and its leading coefficient is positive, you know $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$. This means that the function starts off negative at the left and ends positive at the right.

Step 3: Because each zero listed is the location of a sign change, you can complete the sign chart.

The solutions of $3x^3 - 4x^2 - 13x - 6 \leq 0$ are $x$-values such that $f(x)$ is negative or equal to 0. From the sign chart, you can see that the solution set is $(-\infty, -\frac{3}{5}) \cup \left[-\frac{2}{3}, 3\right]$.

**CHECK**: The graph of $f(x) = 3x^3 - 4x^2 - 13x - 6$ is on or below the $x$-axis on $(-\infty, -\frac{2}{3}) \cup \left[-\frac{2}{3}, 3\right]$.
c. \(x^2 - 10x + 25 > 0\)

The related function \(f(x) = x^2 - 10x + 25\) has one real zero, 5, with multiplicity 2, so the value of \(f(x)\) does not change signs. This function is positive for all real values of \(x\) except \(x = 5\). Therefore, the solution set of \(x^2 - 10x + 25 > 0\) is \((-\infty, 5) \cup (5, \infty)\). The graph of \(f(x)\) supports this conclusion.

d. \(x^2 - 10x + 25 \leq 0\)

The related function \(f(x) = x^2 - 10x + 25\) has a zero at 5. For all other values of \(x\), the function is positive. Therefore, the solution set of \(x^2 - 10x + 25 \leq 0\) is \([5]\).

**Guided Practice**

Solve each inequality.

<table>
<thead>
<tr>
<th>3A. (x^2 + 2x + 5 &gt; 0)</th>
<th>3B. (x^2 + 2x + 5 \leq 0)</th>
<th>3C. (x^2 - 2x - 15 \leq -16)</th>
<th>3D. (x^2 - 2x - 15 &gt; -16)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, \infty))</td>
<td>(\emptyset)</td>
<td>((-\infty, 1) \cup (1, \infty))</td>
<td>((-\infty, \infty))</td>
</tr>
</tbody>
</table>

**Rational Inequalities**

Consider the rational function at the right. Notice the intervals on which \(f(x)\) is positive and negative. While a polynomial function can change signs only at its real zeros, a rational function can change signs at its real zeros or at its points of discontinuity. For this reason, when solving a rational inequality, you must include the zeros of both the numerator and the denominator in your sign chart.

You can begin solving a rational inequality by first writing the inequality in general form with a single rational expression on the left and a zero on the right.

**Example 4: Solve a Rational Inequality**

Solve \(\frac{4}{x - 6} + \frac{2}{x + 1} > 0\).

\[
\begin{align*}
\frac{4}{x - 6} + \frac{2}{x + 1} &> 0 \\
\text{Original inequality} \\
4x + 4 + 2x - 12 &> 0 \\
\text{Use the LCD, } (x - 6)(x + 1), \text{ to rewrite each fraction. Then add.} \\
6x - 8 &> 0 \\
\text{Simplify.} \\
\frac{6x - 8}{(x - 6)(x + 1)} &> 0 \\
\text{Let } f(x) = \frac{6x - 8}{(x - 6)(x + 1)}. \text{ The zeros and undefined points of the inequality are the zeros of the numerator, } \frac{3}{2}, \text{ and denominator, 6 and } -1. \text{ Create a sign chart using these numbers. Then choose test } x\text{-values in each interval to determine if } f(x) \text{ is positive or negative.} \\
f(x) &= \frac{6x - 8}{(x - 6)(x + 1)}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Test</th>
<th>(-)</th>
<th>+(+)</th>
<th>zero</th>
<th>(-)</th>
<th>+(+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty, -1)</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
</tr>
<tr>
<td>(-1, \frac{3}{2})</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
</tr>
<tr>
<td>(\frac{3}{2}, 6)</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
</tr>
<tr>
<td>(6, \infty)</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
<td>zero</td>
<td>under.</td>
</tr>
</tbody>
</table>

The solution set of the original inequality is the union of those intervals for which \(f(x)\) is positive, \((-\infty, -1) \cup (6, \infty)\). The graph of \(f(x) = \frac{4}{x - 6} + \frac{2}{x + 1}\) in Figure 2.6.1 supports this conclusion.

**Tips for New Teachers**

**Changing Signs** A rational function can change signs at its real zeros or at its points of discontinuity, so the zeros of both the numerator and the denominator are included in a sign chart.

**Teach with Tech**

**Document Camera** Choose several students to demonstrate and explain to the class how to use a sign chart to test intervals when solving an inequality.

**Focus on Mathematical Content**

**Solving Rational Inequalities** To solve a rational inequality, first find the zeros from the numerator and the undefined points from the denominator. Use these zeros and undefined points to divide a number line into intervals represented in a sign chart. Choose one number from each interval and evaluate \(f(x)\) to determine if \(f(x)\) is positive or negative in that interval. If \(f(x) < 0\), then the inequality is true when \(f(x)\) is negative. If \(f(x) > 0\), then the inequality is true when \(f(x)\) is positive.

**Extension** Have students determine the positive values of \(n\) for which the cube of \(n\) will be greater than 10 times the square of \(n\). \((10, \infty)\)
A carpenter is making tables. The tables have rectangular tops with a perimeter of 20 feet and an area of at least 24 square feet. Write and solve an inequality that can be used to determine the possible lengths to which the tables can be made.

\( \ell(10 - \ell) \geq 24; \) 4 ft to 6 ft

### Additional Answers

1. \([-4, 2]\)
2. \((-\infty, -1) \cup (6, \infty)\)
3. \((-\infty, -\frac{1}{3}) \cup [8, \infty)\)
4. \((-\infty, \frac{5}{2}) \cup (4, \infty)\)
5. \((-\infty, -\frac{1}{2}) \cup \left(\frac{2}{3}, \infty\right)\)
6. \((-\infty, -2] \cup \left[\frac{1}{2}, 6\right]\)
7. \(-3, -\frac{3}{4}\) \(\cup (0, \infty)\)
8. \(-\frac{2}{5}, 3\) \(\cup (6, \infty)\)
9. \((-\infty, \infty)\)
10. \(\emptyset\)
11. \(\emptyset\)
12. \((-\infty, \infty)\)
13. \(\{4\}\)
14. \(-3\)
15. \(-2\)
16. \((-\infty, \infty)\)
17. \(-0.0004x^2 + 80x - 1,000,000 \geq 2,000,000; [50,000, 150,000]\) or \(50,000 \leq x \leq 150,000\)
18. \(-\frac{15}{2}, -4\)
19. \((-\infty, 5)\)
20. \(\left[\frac{25}{3}, 2\right]\)
21. \((-\infty, -\frac{20}{3}) \cup (-3, \infty)\)
22. \((-\infty, -\frac{2}{5}) \cup \left(-\frac{7}{27}, \infty\right)\)
23. \((-\infty, \frac{14}{13}) \cup \left(\frac{5}{3}, \infty\right)\)
24. \((-\infty, -1) \cup \left[-\frac{3}{4}, 3\right] \cup (4, \infty)\)
25. \(-3, -\frac{3}{2}\) \(\cup (1, \infty)\)
26. \(-5, -4\) \(\cup \left(-4, -\frac{3}{2}\right]\)

### Interpersonal Learners

Have students work together in mixed-ability groups of three to factor the numerator in the rational inequality \(\frac{x^2 - x - 12}{x + 4} \geq 0\). Ask each member of the group to name a different critical number. Then ask groups to explain why the solution can be written as \((-4, -3) \cup \{4, \infty\}\). -3, 4, -4; It is interval notation for \(\{x| -4 < x \leq -3 \text{ or } 4 \leq x, x \in \mathbb{R}\}\).
Solve each inequality. (Examples 1–3) 1–16. See margin.
1. \((x + 4)(x - 2) \leq 0\)
2. \((x - 6)(x + 1) > 0\)
3. \((3x + 1)(x - 8) \geq 0\)
4. \((x - 4)(2x + 5) < 0\)
5. \((4 - 6y)(2y + 1) < 0\)
6. \(2x^2 - 9x^2 - 20x + 12 \leq 0\)
7. \(-8x^3 - 10x^2 - 18x < 0\)
8. \(5x^3 - 43x^2 + 72x + 36 > 0\)
9. \(x^2 + 6x > -10\)
10. \(2x^2 - x - 4\)
11. \(4x^2 + 8x - 5 \leq 2x\)
12. \(2x^2 + 8x + 4x - 8\)
13. \(2b^2 + 16 \leq b^2 + 8b\)
14. \(c^2 + 12 \leq 3 - 6c\)
15. \(-a^2 \geq 4t + 4\)
16. \(3d^2 + 16 \leq d^2 + 16d\)

17. BUSINESS A new company projects that its first-year revenue will be \(r(x) = 120x - 0.0004x^2\) and the startup cost will be \(c(x) = 40x + 1,000,000\), where \(x\) is the number of products sold. The net profit \(p\) that will make the first year's equal to \(p = r - c\). Solve and write an inequality to determine how many products the company must sell to make a profit of at least $2,000,000. (Example 1)

See margin.
Solve each inequality. (Example 4) 18–27. See margin.
18. \(\frac{x - 3}{x + 4} > 3\)
19. \(\frac{x + 6}{x} \leq 1\)
20. \(\frac{2x + 1}{x - 6} \leq 4\)
21. \(\frac{3x - 2}{x + 3} < 6\)
22. \(\frac{3 - 2x}{5x} < 5\)
23. \(\frac{4x + 1}{3x} \leq -3\)
24. \((x + 3)(2x - 3) \leq 6\)
25. \((x + 1)(x - 3) \leq 6\)
26. \((x^2 + 65) \leq 5\)

27. \(2x^2 + 14 \leq 12\)

28. CHARITY The Key Club at a high school is having a dinner as a fundraiser for charity. A dining hall that can accommodate 80 people will cost $1000 to rent. If each ticket costs $20 in advance or $22 the day of the dinner, and the same number of people bought tickets in advance as bought the day of the dinner, write and solve an inequality to determine the minimum number of people that must attend for the club to make a profit of at least $500. (Example 5)

See margin.
Find the domain of each expression. 30–35. See margin.
30. \(\sqrt{x^2 + 5x + 6}\)
31. \(\sqrt{x^2 - 3x - 40}\)
32. \(\sqrt{16 - x^2}\)
33. \(\sqrt{x^2 - 9}\)
34. \(\sqrt{\frac{x}{x^2 - 25}}\)
35. \(\sqrt{\frac{3}{x}}\)

Find the solution set of \(f(x) - g(x) \geq 0\). 36.
37.
38. SALES A vendor sells hot dogs at each school sporting event. The cost of each hot dog is $0.38 and the cost of each bun is $0.12. The vendor rents the hot dog cart that he uses for $1000. If he wants his costs to be less than his profits after selling 400 hot dogs, what should the vendor charge for each hot dog? more than $3

39. PARKS AND RECREATION A rectangular playing field for a community park is to have a perimeter of 112 feet and an area of at least 588 square feet. a–c. See margin.
a. Write an inequality that could be used to find the possible lengths to which the field can be constructed.
b. Solve the inequality found in part a and interpret the solution.
c. How does the inequality and solution change if the area of the field is to be no more than 588 square feet? Interpret the solution in the context of the situation.

Solve each inequality. (Hint: Test every possible solution interval that lies within the domain using the original inequality.) 40–46. See margin.
40. \(\sqrt{9y + 19} - \sqrt{6y - 5} > 3\)
41. \(\sqrt{x + 4} + \sqrt{x - 4} \leq 4\)
42. \(\sqrt{12y + 72} - \sqrt{6y - 11} \geq 7\)
43. \(\sqrt{25 - 12x} - \sqrt{16 - 4x} < 5\)

Determine the inequality shown in each graph.
44.
45.

Solve each inequality.
46. \(2y^2 - 9y^2 - 29y^2 + 60y + 36 > 0\)
47. \(3x^4 + 7x^3 - 56x^2 - 80x < 0\)
48. \(c^3 + 6c^4 - 12c^3 - 56c^2 + 96c \geq 0\)
49. \(3x^5 + 13x^4 - 137x^3 - 353x^2 + 330x + 144 \leq 0\)

3 Practice
Formative Assessment
Use Exercises 1–35 to check for understanding.

Then use the table below to customize assignments for your students.

WatchOut!
Common Error In Exercises 40–43, remind students that when solving inequalities that involve eliminating a radical by raising each side to a power, it is important to test every possible solution interval using the original inequality. While a sign chart can aid in developing a solution set, it may not account for all solutions.

Additional Answers
27. \((-\infty, \frac{13}{6}) \cup (4, \infty)\)
29. \(\frac{750 + 25x}{x} < 120\) for 8 to 14 people
30. \((-\infty, -3) \cup [-2, \infty)\)
31. \((-\infty, -5) \cup [8, \infty)\)
32. \([-4, 4)\)
33. \((-\infty, -3) \cup [3, \infty)\)
34. \((-5, 0) \cup (5, \infty)\)
35. \((-\infty, -6) \cup (-6, 6) \cup (6, \infty)\)
39a. \((56 - \ell) \geq 588\) or \(-\ell^2 + 56\ell - 588 \geq 0\)
39b. \([14, 42]\): The length of the playing field is at least 14 and at most 42 feet.
39c. \(0 \leq \ell(56 - \ell) \leq 588\); \((0, 14) \cup (42, 56)\); Sample answer: The area of the playing field must be greater than 0 square feet but at most 588 square feet, so \(0 \leq \ell \leq 14\) or \(42 \leq \ell \leq 56\).
46. \((-\infty, -3) \cup \left[\frac{-1}{2}, 2\right) \cup (6, \infty)\)
47. \((-5, \frac{4}{3}) \cup (0, 4)\)
48. \([-6, -4) \cup [0, \infty)\)
49. \((-\infty, -8) \cup \left[-3, \frac{-1}{3}\right] \cup [1, 6]\)
50. **Packaging** A company sells cylindrical oil containers like the one shown. a–c. See margin.

![Cylindrical Oil Container](image)

- Use the volume of the container to express its surface area as a function of its radius in centimeters. (Hint: 1 liter = 1000 cubic centimeters)
- The company wants the surface area of the container to be less than 2400 square centimeters. Write an inequality that could be used to find the possible radii to meet this requirement.
- Use a graphing calculator to solve the inequality you wrote in part b and interpret the solution.

51. Solve each inequality.

- \((\infty, -3) \cup (-3, -\frac{1}{2}) \cup \left(\frac{-1}{2}, 4\right)\)
- \((6, \infty) \cup [-1, \infty)\)
- \((-\infty, -6) \cup \left[\frac{4}{3}, 3\right) \cup (0, \infty)\)

52. **Study Time** Jarrick determines that with the information that he currently knows, he can achieve a score of a 75% on his test. Jarrick believes that for every 5 complete minutes he spends studying, he will raise his score by 1%.

a. If Jarrick wants to obtain a score of at least 89.5%, write an inequality that could be used to find the time t that he will have to spend studying.

b. Solve the inequality that you wrote in part a and interpret the solution. 75% sample answer: Since \(t \geq 72.5\), Jarrick will have to spend 75 minutes studying for the test.

c. **Games** A skee ball machine pays out 3 tickets each time a person plays and then 2 additional tickets for every 80 points the player scores. \(f(x) = 2\left(\frac{x}{80}\right) + 3\)

a. Write a nonlinear function to model the amount of tickets received for an x-point score.

b. Write an inequality that could be used to find the score a player would need in order to receive at least 11 tickets.

53. **Writing in Math** Explain why you cannot solve \(\frac{3x + 1}{x - 2} < 6\) by multiplying each side by \(x - 2\). 55. If \(k\) is nonnegative, find the interval for \(x\) for which each inequality is true. 58–61. See margin.

54. **Word Problem Practice** On the previous page, you created a sign chart using the real zeros. Create a sign chart using these real zeros.

55. **Study Guide and Intervention** Nonlinear Inequalities

- Add or subtract to solve each nonlinear inequality.
- Multiply or divide to solve each nonlinear inequality.
- Use the addition property of inequalities to solve each nonlinear inequality.
- Use the multiplication property of inequalities to solve each nonlinear inequality.

56. **Multiple Representations** In this problem, you will investigate absolute value nonlinear inequalities.

a. **Tabular** Copy and complete the table below.

<table>
<thead>
<tr>
<th>Function</th>
<th>Zeros</th>
<th>Undefined Points</th>
<th>b–d. See Chapter 2 Answer Appendix</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = \frac{x - 1}{x + 2})</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>(g(x) = \frac{2x - 5}{x - 3})</td>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(h(x) = \frac{4x - 8}{x + 11})</td>
<td>-4</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

57. **H.O.T. Problems** Use Higher-Order Thinking Skills

63. **Error Analysis** Ajay and Mae are solving \(\frac{x}{3} + \frac{2}{3} \geq 0\).

Ajay thinks that the solution is \((-\infty, 0)\) or \([0, \infty)\), and Mae thinks that the solution is \((-\infty, \infty)\). Is either of them correct? Explain your reasoning. **See margin.**

64. **Reasoning** If the solution set of a polynomial inequality is \((-\infty, 3)\), what will be the solution set if the inequality symbol is reversed? Explain your reasoning. **See margin.**

65. **Challenge** Determine the values for which \((a + b)^2 > (c + d)^2\) if \(a < b < c < d\). **See margin.**

66. **Reasoning** If \(0 < c < d\), find the interval on which \((x - c)(x - d) \leq 0\) is true. **See margin.**

67. **Challenge** What is the solution set of \((x - a)^2n > 0\) if \(a\) is a natural number? \((-\infty, a) \cup (a, \infty)\) **See margin.**

68. **Reasoning** What happens to the solution set of \((x + a)(x - b) < 0\) if the expression is changed to \((x + a)(x - b) < 0\), where \(a\) and \(b\) are greater than zero? **See margin.**

69. **Writing in Math** Explain why you cannot solve \(\frac{3x + 1}{x - 2} < 6\) by multiplying each side by \(x - 2\). **See margin.**

### Additional Answers

50a. \(S(r) = 2\pi r^2 + \frac{4000}{r}\)

50b. \(2\pi r^2 + \frac{4000}{r} < 2400\) or \(2mr^2 + \frac{4000}{r} < 2400 < 0\)

50c. \((-20.33, 0) \cup (1.68, 18.65)\); Because the radius cannot be negative, the only possible lengths for the radius are between 1.68 cm and 18.65 cm.
a. Use a graphing calculator to model the data using a polynomial function with a degree of 3.

b. Divide using long division.

c. Use the model to estimate the closing price of the stock on day 25.

Day Price(s) Day Price(s)
1 36.15 15 15.64
5 27.91 29 19.38
7 26.10 21 9.56
10 22.37 28 9.95
12 19.61 30 12.25

Sample answer: The zeros will remain the same but the solution set will consist of all numbers outside of the original solution set.

66. \([c, d]\); Sample answer: Because \(c < d\), when \(x < c\), \(x < d\), so \((x - c)\) and \((x - d)\) will be negative and \((x - c)(x - d)\) will be positive. When \(x > d\), both factors will be positive, so \((x - c)(x - d)\) will be positive. When \(c \leq x \leq d\), \((x - c)\) will be positive or zero, \((x - d)\) will be negative or zero, and \((x - c)(x - d)\) will be either negative or zero.

68. Sample answer: The zeros will remain the same but the solution set will consist of all numbers outside of the original solution set.

For example, the solution set for the original inequality is \((-a, b)\), while the solution set for the new inequality is \((-\infty, -a) \cup (b, \infty)\).

69. Sample answer: There will be instances when \(x - 2\) is a negative value. When this occurs, we are multiplying an inequality by a negative and the inequality symbol will need to be reversed. The problem occurs here because \(x - 2\) could be positive or negative.
## Chapter Summary

### Key Concepts

#### Power and Radical Functions (Lesson 2-1)
- A power function is any function of the form \( f(x) = ax^n \), where \( a \) and \( n \) are nonzero real numbers.
- A monomial function is any function that can be written as \( f(x) = a \) or \( f(x) = ax^n \), where \( a \) and \( n \) are nonzero constant real numbers.
- A radical function is a function that can be written as \( f(x) = \sqrt[p]{x} \), where \( n \) and \( p \) are positive integers greater than 1 that have no common factors.

#### Polynomial Functions (Lesson 2-2)
- A polynomial function is any function of the form \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \), where \( a_n \neq 0 \). The degree is \( n \).
- The graph of a polynomial function has at most \( n \) distinct real zeros and at most \( n - 1 \) turning points.
- The behavior of a polynomial graph at its zero \( c \) depends on the multiplicity of the factor \( (x - c) \).

#### The Remainder and Factor Theorems (Lesson 2-3)
- Synthetic division is a shortcut for dividing a polynomial by a linear factor of the form \( x - c \).
- If a polynomial \( f \) is divided by \( x - c \), the remainder is equal to \( f(c) \).
- \( x - c \) is a factor of a polynomial \( f \) if and only if \( f(c) = 0 \).

#### Zeros of Polynomial Functions (Lesson 2-4)
- If \( f(x) = a_nx^n + \ldots + a_1x + a_0 \) is a function with integer coefficients, then any rational zero of \( f(x) \) is of the form \( \frac{p}{q} \), where \( p \) and \( q \) have no common factors, \( p \) is a factor of \( a_0 \), and \( q \) is a factor of \( a_n \).
- A polynomial of degree \( n \) has \( n \) zeros, including repeated zeros, in the complex system. It also has \( n \) factors:
  \[ f(x) = a_n(x - c_0)(x - c_1)(x - c_2) \ldots (x - c_n). \]

#### Rational Functions (Lesson 2-5)
- The graph of a rational function \( f \) has a vertical asymptote \( x = c \) if \( \lim_{x \to c^-} f(x) = \pm \infty \) or \( \lim_{x \to c^+} f(x) = \pm \infty \).
- The graph of a rational function \( f \) has a horizontal asymptote \( y = c \) if \( \lim_{x \to \pm \infty} f(x) = c \).
- A rational function \( f(x) = \frac{a(x)}{b(x)} \) may have vertical asymptotes, horizontal asymptotes, or oblique asymptotes, \( x \)-intercepts, and \( y \)-intercepts. They can all be determined algebraically.

#### Nonlinear Inequalities (Lesson 2-6)
- The sign chart for a rational inequality must include zeros and undefined points.

### Key Vocabulary

- **complex conjugates** (p. 124)
- **extraneous solution** (p. 91)
- **horizontal asymptote** (p. 131)
- **irreducible over the reals** (p. 124)
- **leading coefficient** (p. 97)
- **leading-term test** (p. 98)
- **lower bound** (p. 121)
- **multiplicity** (p. 102)
- **oblique asymptote** (p. 134)
- **polynomial function** (p. 97)
- **power function** (p. 86)
- **quartic function** (p. 99)
- **rational function** (p. 130)
- **repeated zero** (p. 101)
- **sign chart** (p. 141)
- **synthetic division** (p. 111)
- **synthetic substitution** (p. 113)
- **turning point** (p. 99)
- **upper bound** (p. 121)
- **vertical asymptote** (p. 131)

### Vocabulary Check

Identify the word or phrase that best completes each sentence.

1. The coefficient of the term with the greatest exponent of the variable is the \( \text{leading coefficient} \).
2. A (polynomial function, power function) is a function of the form \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \), where \( a_n \), \( a_{n-1} \), \( \ldots, a_1 \) are real numbers and \( n \) is a natural number. \( \text{polynomial function} \).
3. A function that has multiple factors of \( x - c \) has \( \text{repeated zeros} \), turning points.
4. (Polynomial division, Synthetic division) is a short way to divide polynomials by linear factors. \( \text{Synthetic division} \).
5. The (Remainder Theorem, Factor Theorem) relates the linear factors of a polynomial with the zeros of its related function. \( \text{Factor Theorem} \).
6. Some of the possible zeros for a polynomial function can be listed using the (Factor, Rational Zero) Theorems. \( \text{Rational Zeros} \).
7. (Vertical, Horizontal) asymptotes are determined by the zeros of the denominator of a rational function. \( \text{Vertical} \).
8. The zeros of the (denominator, numerator) determine the \( x \)-intercepts of the graph of a rational function. \( \text{numerator} \).
9. (Horizontal, Oblique) asymptotes occur when a rational function has a denominator with a degree greater than 0 and a numerator with degree one greater than its denominator. \( \text{Oblique} \).
10. A (quartic function, power function) is a function of the form \( f(x) = ax^n \), where \( a \) and \( n \) are nonzero constant real numbers.

### Additional Answers

21. \( \lim_{x \to -\infty} f(x) = -\infty; \lim_{x \to \infty} f(x) = -\infty \); The degree is even and the leading coefficient is negative.
22. \( \lim_{x \to -\infty} f(x) = -\infty; \lim_{x \to \infty} f(x) = \infty \); The degree is odd and the leading coefficient is positive.
23. \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \); The degree is odd and the leading coefficient is positive.
24. \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \); The degree is odd and the leading coefficient is positive.
25. 3 real zeros and 2 turning points; 0, 3, and 4
26. 5 real zeros and 4 turning points; –10, 0, and 2
27. 4 real zeros and 3 turning points; –3, –1, 1, and 3
28. 4 real zeros and 3 turning points; \( \sqrt{5} \) and \( -\sqrt{5} \)
Lesson-by-Lesson Review

2.1 Power and Radical Functions

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing. See Chapter 2 Answer Appendix.

11. \( f(x) = 5x^6 \)
12. \( f(x) = -8x^3 \)
13. \( f(x) = x^{-3} \)
14. \( f(x) = \frac{3}{4}x^{-4} \)
15. \( f(x) = \sqrt[3]{5x - 6} - 11 \)
16. \( f(x) = -\frac{3}{4}\sqrt[3]{6x^2} - 1 + 2 \)

Solve each equation.
17. \( 2x = 4 + \sqrt[3]{7x - 12} \)
18. \( \sqrt[3]{4x + 5} + 1 = 4x \)
19. \( 4 = \sqrt[3]{8x + 1} - \sqrt[3]{17} - 4x \)
20. \( \sqrt{x^2 + 3} - 1 = 3 \)

Example 1

Graph and analyze \( f(x) = -4x^{-5} \). Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.016</td>
</tr>
<tr>
<td>-2</td>
<td>0.125</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>undefined</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-0.125</td>
</tr>
<tr>
<td>3</td>
<td>-0.016</td>
</tr>
</tbody>
</table>

Domain: \( (-\infty, 0) \cup (0, \infty) \)
Range: \( (-\infty, 0) \cup (0, \infty) \)
Intercepts: none
End behavior: \( \lim_{x \to -\infty} f(x) = 0 \) and \( \lim_{x \to \infty} f(x) = 0 \)
Continuity: infinite discontinuity at \( x = 0 \)
Increasing: \( (-\infty, 0) \)
Increasing: \( (0, \infty) \)

2.2 Polynomial Functions

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test. See margin.

21–24. See margin.

25. \( f(x) = x^3 - 7x + 12 \)
26. \( f(x) = x^3 + 8x^2 - 20x^2 \)
27. \( f(x) = x^4 - 10x^2 + 9 \)
28. \( f(x) = x^4 - 25 \)

For each function, (a) apply the leading term test, (b) find the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function. See margin.

29. \( f(x) = x^3(x - 3)(x + 4)^2 \)
30. \( f(x) = (x - 5)^2(x - 1)^2 \)

Example 2

Describe the end behavior of the graph of \( f(x) = -2x^5 + 3x^3 - 8x^2 - 6 \) using limits. Explain your reasoning using the leading term test.

The degree is 5 and the leading coefficient is negative. Because the degree is odd and the leading coefficient is negative, \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = -\infty \).

Example 3

State the number of possible real zeros and turning points for \( f(x) = x^3 + 6x^2 + 9x \). Then find all the real zeros by factoring.

The degree of \( f(x) \) is 3, so \( f(x) \) has at most 3 distinct real zeros and at most 3 – 1 = 2 turning points. To find the real zeros, solve the related equation \( f(x) = 0 \) by factoring.

\[
x^3 + 6x^2 + 9x = x(x^2 + 6x + 9) = x(x + 3)(x + 3) \text{ or } x(x + 3)^2
\]

The expression has 3 factors but only 2 distinct real zeros, 0 and -3.
**Additional Answers**

50. \( D = \{ x \mid x \neq -4, x \in \mathbb{R} \} \);  
\( x = -4 \)

51. \( D = \{ x \mid x \neq 5, -5, x \in \mathbb{R} \} \);  
\( x = 5, x = -5, y = 1 \)

52. \( D = \{ x \mid x \neq 5, -3, x \in \mathbb{R} \} \);  
\( x = 5, x = -3, y = 0 \)

53. \( D = \{ x \mid x \neq -3, -9, x \in \mathbb{R} \} \);  
\( x = -3, x = -9, y = 1 \)

54. asymptotes: \( x = 5, y = 1 \);  
x-intercept: 0; y-intercept: 0;  
\( D = \{ x \mid x \neq 5, x \in \mathbb{R} \} \)

55. asymptotes: \( x = -4, y = 1 \);  
x-intercept: 2; y-intercept: \(-\frac{1}{2}\);  
\( D = \{ x \mid x \neq -4, x \in \mathbb{R} \} \)

---

**2.3 The Remainder and Factor Theorems**

Divide using long division.

31. \( (x^3 + 8x^2 - 5) \div (x - 2) \)  
\( x^2 + 10x + 20 + \frac{35}{x - 2} \)

32. \( -(3x^3 + 5x^2 - 22x + 5) \div (x^2 + 4) \)  
\(-3x + 5 - \frac{10x + 15}{x^2 + 4} \)

33. \( (2x^3 + 5x^2 - 5x^2 + x^2 - 18x + 10) \div (2x - 1) \)  
\( x^4 + 3x^3 - x^2 - 9 + \frac{1}{2x - 1} \)

Divide using synthetic division.

34. \( (x^3 - 8x^2 + 7x - 15) \div (x - 1) \)  
x^2 - 7x + 15

35. \( (x^3 + 7x^2 - 9x - 18) \div (x - 2) \)  
x^2 + x^2 + 9x + 9

36. \( (2x^3 + 3x^2 - 10x + 16x - 6) \div (2x - 1) \)  
x^2 + 2x^2 - 4x + 6

Use the Factor Theorem to determine if the binomials are factors of \( f(x) \). Use the binomials that are factors to write a factored form of \( f(x) \).

yes; \( f(x) = (x + 3)(x^2 - 8) \)

37. \( f(x) = x^3 + 3x^2 - 8x - 24 \)  
\( (x + 3) \)

38. \( f(x) = 2x^4 - 9x^2 + 2x^2 + 9x - 4 \)  
\( (x - 1), (x + 1) \)

39. \( f(x) = x^4 - 2x^3 - 3x^2 + 4x - 4 \)  
\( (x + 2) \)

**Example 4**

Divide \( (2x^3 - 3x^2 + 5x - 4) \div (2x - 1) \) using synthetic division.

\( \frac{2x^3 - 3x^2 + 5x - 4}{2x - 1} = \frac{2x^2 - 3x^2 + 5x - 2}{2x - 1} = \frac{x^2 - \frac{3}{2}x^2 + \frac{5}{2}x - 2}{x - 1} \)

Therefore, \( c = \frac{1}{2} \). Perform synthetic division.

**Example 5**

Solve \( x^2 + 2x - 16 = 32 = 0 \).

Because the leading coefficient is 1, the possible rational zeros are the factors of \(-32\). The possible rational zeros are \( \pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \) and \( \pm 32 \). Using synthetic substitution, you can determine that \(-2\) is a rational zero.

\( -2 \)  
\( 1 \)  
\( 2 \)  
\( 16 \)  
\( -32 \)

\( -2 \)  
\( 0 \)  
\( 32 \)

\( 1 \)  
\( 0 \)  
\( -16 \)

\( 0 \)

Therefore, \( f(x) = (x + 2)(x^2 - 16) \). This polynomial can be written \( f(x) = (x + 2)(x + 4)(x - 4) \). The rational zeros of \( f(x) \) are \(-2, 4, \) and \(-4\).
2-5 Rational Functions

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any. 50–53. See margin.

50. \( f(x) = \frac{x^2 - 1}{x + 4} \)
51. \( f(x) = \frac{x^2}{x^2 - 25} \)
52. \( f(x) = \frac{x(x - 3)}{(x - 5)(x + 3)^2} \)
53. \( f(x) = \frac{(x - 5)(x - 2)}{(x + 3)(x + 9)} \)

For each function, determine any asymptotes and intercepts. Then graph the function, and state its domain. 54–59. See margin.

54. \( f(x) = \frac{x}{x - 5} \)
55. \( f(x) = \frac{x - 2}{x + 4} \)
56. \( f(x) = \frac{x + 3(x - 4)}{x + 5(x - 6)} \)
57. \( f(x) = \frac{x(x + 7)}{x(x + 6)(x - 3)} \)
58. \( f(x) = \frac{x + 2}{x^2 - 1} \)
59. \( f(x) = \frac{x^2 - 16}{x^2 - 6x + 5} \)

Solve each equation.

60. \( \frac{12}{x} + x - 8 = 1 \)
61. \( \frac{2}{x} + \frac{3}{x} = \frac{-6}{x} \)
62. \( \frac{2}{d + 4} = \frac{2}{d^2 + 3d - 4} \)
63. \( \frac{1}{n - 2} = \frac{2n + 1}{n^2 + 2n - 8} + \frac{7}{n + 4} \)

Example 6

Find the domain of \( f(x) = \frac{x + 7}{x + 1} \) and any vertical or horizontal asymptotes.

**Step 1** Find the domain.

The function is undefined at the zero of the denominator \( h(x) = x + 1 \), which is \(-1\). The domain of \( f \) is all real numbers except \( x = -1 \).

**Step 2** Find the asymptotes, if any.

Check for vertical asymptotes.

The zero of the denominator is \(-1\), so there is a vertical asymptote at \( x = -1 \).

Check for horizontal asymptotes.

The degree of the numerator is less than the degree of the denominator. The ratio of the leading coefficient is \( \frac{1}{1} = 1 \).

Therefore, \( y = 1 \) is a horizontal asymptote.

2-6 Nonlinear Inequalities (pp. 141–147)

Solve each inequality.

64. \( (x + 5)(x - 3) \leq 0 \)
65. \( x^2 - 6x - 16 > 0 \)
66. \( x^2 + 5x^2 \leq 0 \)
67. \( 2x^2 + 13x + 15 < 0 \)
68. \( x^2 + 12x + 36 \leq 0 \)
69. \( x^2 + 4 < 0 \)
70. \( x^2 + 4x + 4 > 0 \)
71. \( \frac{x - 5}{x} < 0 \)
72. \( \frac{2x + 6}{3(x + 2)} \geq 0 \)

\( \left( -\infty, -\frac{4}{3} \right) \cup \left( -\frac{1}{2}, \infty \right) \)

Example 7

Solve \( x^2 + 5x^2 - 36x \leq 0 \).

Factoring the polynomial \( f(x) = x^2 + 5x^2 - 36x \) yields \( f(x) = x(x + 9)(x - 4) \), so \( f(x) \) has real zeros at \( 0 \), \(-9 \), and \( 4 \).

Create a sign chart using these zeros. Then substitute an \( x \)-value from each test interval into the function to determine whether \( f(x) \) is positive or negative at that point.

\( (-\infty, -9) \cup (-9, 4) \cup (4, \infty) \)

Because \( f(x) \) is negative on the first and third intervals, the solution of \( x^2 + 5x^2 - 36x \leq 0 \) is \( (-\infty, -9) \cup (4, \infty) \).

59. asymptotes: \( x = 0 \), \( x = 1 \), \( x = 5 \), \( y = 0 \);
 \( x \)-intercepts: 4 and -4; \( D = \{ x \mid x \neq 0, 1, 5, x \in \mathbb{R} \} \)

Additional Answers

56. asymptotes: \( x = -5 \), \( x = 6 \), \( y = 1 \);
 \( x \)-intercepts: \(-3 \) and \( 4 \);
 \( y \)-intercept: \( \frac{2}{3} \); \( D = \{ x \mid x \neq -5, 6, x \in \mathbb{R} \} \)

57. asymptotes: \( x = -6 \), \( x = 3 \), \( y = 1 \);
 \( x \)-intercepts: \( 0 \) and \(-7 \);
 \( y \)-intercept: \( 0 \); \( D = \{ x \mid x \neq -6, 3, x \in \mathbb{R} \} \)

58. asymptotes: \( x = -1 \), \( x = 1 \), \( y = 0 \);
 \( x \)-intercept: \(-2 \); \( y \)-intercept: \(-2 \);
 \( D = \{ x \mid x \neq -1, 1, x \in \mathbb{R} \} \)
Applications and Problem Solving

74. PHYSICS Kepler’s Third Law of Planetary Motion implies that the time T it takes for a planet to complete one revolution in its orbit about the Sun is given by \( T = R^3 \), where \( R \) is the planet’s mean distance from the Sun. Time is measured in Earth years, and distance is measured in astronomical units. (Lesson 2-1)
   a. State the relevant domain and range of the function.
   b. Graph the function. See margin.
   c. The time for Mars to orbit the Sun is observed to be 1.88 Earth years. Determine Mars’ average distance from the Sun in miles, given that one astronomical unit equals 93 million miles.

   \[ \text{Sample answer: about 141.66 million mi} \]

75. PUMPKIN LAUNCH Mr. Roberts’ technology class constructed a catapult to compete in the county’s annual pumpkin launch. The speed \( v \) in miles per hour of a launched pumpkin after \( t \) seconds is given.

   \[ f(t) = 43.63t^{-0.12} \]
   a. Create a scatter plot of the data.
   b. Determine a power function to model the data.
   c. Use the function to predict the speed at which a pumpkin is traveling after 1.2 seconds. about 35.6 mi/h
   d. Use the function to predict the time at which the pumpkin’s speed is 47 miles per hour. about 0.94 second

76. AMUSEMENT PARKS The elevation above the ground for a rider on the Big Monster roller coaster is given in the table. (Lesson 2-2)

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elevation (ft)</td>
<td>85</td>
<td>62</td>
<td>44</td>
<td>17</td>
</tr>
</tbody>
</table>

   a. Create a scatter plot of the data and determine the type of polynomial function that could be used to represent the data.
   b. Write a polynomial function to model the data set. Round each coefficient to the nearest thousandth and state the correlation coefficient. See margin.
   c. The model to determine a rider’s elevation at 17 seconds.
   d. Use the model to determine approximately the first time a rider is 50 feet above the ground.

   \[ \text{Sample answer: about 11.4 seconds} \]

77. GARDENING Mark’s parents seeded their new lawn in 2001. From 2001 until 2011, the amount of crab grass increased following the model \( f(x) = 0.032x^2 - 1.171x^2 + 6.943x + 76 \); \( r^2 = 0.997 \)

\[ f(x) = -50x^2 + 500x + 10,000 \]

78. BUSINESS A used bookstore sells an average of 1000 books each month at an average price of $10 per book. Due to rising costs the owner wants to raise the price of all books. She figures she will sell 50 fewer books for every $1 she raises the prices. (Lesson 2-4)
   a. Write a function for her total sales after raising the price of her books $x$ dollars. See margin.
   b. How many dollars does she need to raise the price of her books so that the total amount of sales is $11,250? $5
   c. What is the maximum amount that she can increase prices and still achieve $10,000 in total sales? Explain. See margin.

79. AGRICULTURE A farmer wants to make a rectangular enclosure using one side of her barn and 80 meters of fence material. Determine the dimensions of the enclosure. Assume that the width of the enclosure will not be greater than the side of the barn. (Lesson 2-4)

80. ENVIRONMENT A pond is know to contain 0.40% acid. The pond contains 50,000 gallons of water. (Lesson 2-5)
   a. How many gallons of acid are in the pond? 200 gal
   b. Suppose x gallons of pure water was added to the pond. Write \( p(x) \), the percentage of acid in the pond after x gallons of pure water are added. See margin.
   c. Find the horizontal asymptote of \( p(x) \).
   d. Does the function have any vertical asymptotes? Explain. No; sample answer: \( p(x) \) is defined for all numbers in the interval \([0, \infty)\).

81. BUSINESS For selling x cakes, a baker will make \( b(x) = x^2 - 5x - 150 \) hundreds of dollars in revenue. Determine the minimum number of cakes the baker needs to sell in order to make a profit. (Lesson 2-5) 18 cakes

82. DANCE The junior class would like to organize a school dance as a fundraiser. A hall that the class wants to rent costs $300 plus an additional charge of $5 per person. (Lesson 2-6)
   a. Write and solve an inequality to determine how many people need to attend the dance if the junior class would like to keep the cost per person under $10.
   b. The hall will provide a DJ for an extra $100. How many people would have to attend the dance to keep the cost under $10 per person?
Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.  

1–2. See margin.

1. \( f(x) = 0.25x^{-3} \)

2. \( f(x) = 8x^3 \)

Solve each equation.

3. \( x = \sqrt{4 - x} - 8 \)  
   \(-5\)

4. \( \sqrt{5x + 4} = \sqrt{9 - x} + 7 \)

9.

5. \( -2 + \sqrt{3x + 2} = x \)

6. \( 5b - \sqrt{7a^2 + 4} = 54 \)

50

7. \( 7x^3 - 5x^2 - 14x^0 = 0 \)

8. \( x^3 - 3x^2 - 10x = -24 \)

6. \(-6\)

\( 0, 7, -2 \)

\( x \)

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test.

9. \( f(x) = 5x^3 - 3x^2 - 7x^2 + 11x - 8 \)  
   \( 9\)

10. \( f(x) = -3x^3 - 8x^2 + 7x^2 + 5 \)

9–10. See margin.

State the number of possible real zeros and turning points of each function. Then find all of the real zeros by factoring.

11. \( f(x) = 4x^4 - 8x^3 - 60x \)

12. \( f(x) = x^5 - 10x \)

13. \( f(x) = x^4 - 11x^2 \)

14. \( f(x) = x^4 - 2x^3 + 23x^2 + 50x - 50 \)

Write a polynomial function of least degree with real coefficients in standard form that has the given zeros.

15. \(-1, 4, -\sqrt{5} \)

16. \(5 - 5, 1, -i \)

17. \( f(x) = x^4 - 3x^3 - 7x^2 + 9x + 12 \)

18. \( f(x) = x^4 - 3x^3 - 7x^2 + 9x + 12 \)

MULTIPLE CHOICE Which function graphed below must have imaginary zeros? J

19. WEATHER The table shows the average high temperature in Bay Town each month.

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp</td>
<td>67.2°</td>
<td>72.5°</td>
<td>67.3°</td>
<td>79.1°</td>
<td>85.5°</td>
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<td>64.8°</td>
</tr>
</tbody>
</table>

20. \( -1, 4, -\sqrt{5} \)

21. \( 5, -5, 1, -i \)

22. \( f(x) = x^4 - 3x^3 - 7x^2 + 9x + 12 \)

a. Make a scatter plot for the data. Answer Appendix.

b. Use a graphing calculator to model the data using a polynomial function with a degree of 3. Use \(x = 1\) for January and round each coefficient to the nearest thousandth.

c. Use the model to predict the average high temperature for the following January. Let \(x = 13\).

1–2. See margin.

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20. \( -1, 4, -\sqrt{5} \)

21. \( 5, -5, 1, -i \)

22. \( f(x) = x^4 - 3x^3 - 7x^2 + 9x + 12 \)

MULTIPLE CHOICE Which function graphed below must have imaginary zeros? J

For each function, (a) apply the leading term test, (b) find the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and (d) graph the function. 15–18. See Chapter 2 Answer Appendix.

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</tbody>
</table>
1 Focus

Objective Approximate the area between a curve and the x-axis.

Teaching Tip
For Activity 1, you may wish to have students use a compass to draw the graph. The center of the semi-circle is at (4, 0). The radius is 4. The bottom left vertices of the four rectangles are (0, 0), (2, 0), (4, 0), and (6, 0). The upper left vertices of the rectangles are (0, f(0)), (2, f(2)), (4, f(4)), and (6, f(6)). The width of each rectangle is 2. The lengths are f(0), f(2), f(4), and f(6).

2 Teach

Working in Cooperative Groups
For Activity 1, pair students with different abilities. Have students work through Steps 1–3 and then answer Analyze the Results Exercises 1–4.

Ask:
- What is the formula for the area of a circle? \( A = \pi r^2 \)
- What is the formula for the area of a semi-circle? \( A = \frac{1}{2} \pi r^2 \)

Additional Answers
2. Sample answer: The extra area found outside the curve that is included in the sum helps to account for the area found under the curve that was not included.
3. 25.13 units\(^2\); Sample answer: The approximation is less than the actual area. As more rectangles are used, the approximation approaches the actual area.
4. Sample answer: Using rectangles of a smaller width should result in a more accurate approximation. Smaller rectangles would better fit the shape of the curve and would help to reduce areas that are not being accounted for.

Activity 1 Approximate Area Under a Curve

Approximate the area between the curve \( f(x) = \sqrt{2-x^2} + 8x \) and the x-axis using rectangles.

Step 1 Draw 4 rectangles with a width of 2 units between \( f(0) \) and the x-axis. The height of the rectangle should be determined when the left endpoint of the rectangle intersects \( f(x) \), as shown in the figure. Notice that the first rectangle will have a height of \( f(0) \) or 0.

Step 2 Calculate the area of each rectangle.

Step 3 Approximate the area of the region by taking the sum of the areas of the rectangles.

Analyze the Results
1. What is the approximation for the area? 21.86 units\(^2\)
2. How does the area of a rectangle that lies outside the graph affect the approximation? See margin.
3. Calculate the actual area of the semicircle. How does the approximation compare to the actual area? See margin.
4. How can rectangles be used to find a more accurate approximation? Explain your reasoning. See margin.

Using relatively large rectangles to estimate the area under a curve may not produce an approximation that is as accurate as desired. Significant sections of area under the curve may go unaccounted for. Similarly, if the rectangles extend beyond the curve, substantial amounts of areas that lie above a curve may be included in the approximation.

In addition, regions are also not always bound by a curve intersecting the x-axis. You have studied many functions with graphs that have different end behaviors. These graphs do not necessarily have two x-intercepts that allow for obvious start and finish points. In those cases, we often estimate the area under the curve for an x-axis interval.

6. Sample answer: There is no extra area that lies above the curve that is included in the approximation that can account for the area that lies under the curve that has not been included. Therefore, the approximation should be lower than the actual area.

7. Yes; Sample answer: For this example, using right endpoints for the rectangles would produce rectangles that overlap the curve and include area that lies above the curve. This would produce a higher approximation. However, this is not always the case. If the curve is symmetrical, like the previous example, the approximations will be the same regardless of the endpoint used.
Activity 2 Approximate Area Under a Curve

Approximate the area between the curve \( f(x) = x^2 + 2 \) and the \( x \)-axis on the interval \([1, 5]\) using rectangles.

Step 1 Draw 4 rectangles with a width of 1 unit between the \( f(x) \) and the \( x \)-axis on the interval \([1, 5]\), as shown in the figure. Use the left endpoint of each sub interval to determine the height of each rectangle.

Step 2 Calculate the area of each rectangle.

Step 3 Approximate the area of the region by determining the sum of the areas of the rectangles.

Step 4 Repeat Steps 1–3 using 8 rectangles, each with a width of 0.5 unit, and 16 rectangles, each with a width of 0.25 unit.

Analyze the Results

5. What value for total area are the approximations approaching? Sample answer: 48 units\(^2\)

6. Using left endpoints, all of the rectangles completely lie under the curve. How does this affect the approximation for the area of the region? See margin.

7. Would the approximations differ if each rectangle’s height was determined by its right endpoint? Is this always true? Explain your reasoning. See margin.

8. What would happen to the approximations if we continued to increase the number of rectangles being used? Explain your reasoning. See margin.

9. Make a conjecture about the relationship between the area under a curve and the number of rectangles used to find the approximation. Explain your answer. See margin.

Model and Apply

10a. 6 rectangles: 280 units\(^2\); 12 rectangles: 286 units\(^2\); 24 rectangles: 287.5 units\(^2\)

10b. Sample answer: Each approximation that is found using more rectangles should be a better representation of the actual area. Smaller rectangles would fill the curved region better and would help to insure that most of the area under the curve is included in the approximation.

9. Sample answer: The more rectangles that are used, the better the approximation of the area. Smaller rectangles fit the desired region better than larger rectangles, thus producing more accurate approximations.
15. \( \{ x \mid x \leq 6, x \in \mathbb{R} \}; (-\infty, 6] \)
16. \( \{ x \mid -2 \leq x, x \in \mathbb{Z} \} \)
17. \( \{ x \mid -2 < x < 9, x \in \mathbb{R} \}; (-2, 9) \)
18. \( \{ x \mid 1 < x \leq 4, x \in \mathbb{R} \}; (1, 4] \)
19. \( \{ x \mid x < -4 \text{ or } x > 5, x \in \mathbb{R} \}; (-\infty, -4) \cup (5, \infty) \)
20. \( \{ x \mid x < -1 \text{ or } x \geq 7, x \in \mathbb{R} \}; (-\infty, -1) \cup [7, \infty) \)

**Pages 87–89, Lesson 2-1 (Guided Practice)**

1A. \( f(x) = 3x^2 \)
\( D = (-\infty, \infty), R = [0, \infty); \) intercept 0; \( \lim_{x \to -\infty} f(x) = \infty \)
and \( \lim_{x \to \infty} f(x) = \infty; \) continuous for all real numbers; decreasing: \((-\infty, 0); \) increasing: \((0, \infty)\)

4A. \([0, 100] \text{ scl: 10} \text{ by } [0, 250] \text{ scl: 25}\)
3. $-x + g = -5x - 8y$ drawn; $D = (-\infty, \infty)$, $R = [0, \infty)$; intercept: 0; $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$; continuous for all real numbers; decreasing: $(-\infty, 0)$; increasing: $(0, \infty)$

4. $D = (-\infty, \infty)$, $R = (-\infty, \infty)$; intercept: 0; $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$; continuous for all real numbers; decreasing: $(-\infty, \infty)$

5. $D = (-\infty, \infty)$, $R = (-\infty, \infty)$; intercept: 0; $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$; continuous for all real numbers; increasing: $(-\infty, \infty)$

6. $D = (-\infty, \infty)$, $R = [0, \infty)$; intercept: 0; $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$; continuous for all real numbers; decreasing: $(-\infty, 0)$; increasing: $(0, \infty)$

7. $D = (-\infty, \infty), R = (-\infty, \infty)$; intercept: 0; $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$; continuous for all real numbers; decreasing: $(-\infty, \infty)$

8. $D = (-\infty, \infty), R = (-\infty, 0)$; intercept: 0; $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$; continuous for all real numbers; increasing: $(-\infty, 0)$; decreasing: $(0, \infty)$

9. $D = (-\infty, 0) \cup (0, \infty)$, $R = (0, \infty)$; no intercepts; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$; decreasing: $(0, \infty)$

10. $D = (-\infty, 0) \cup (0, \infty)$, $R = (-\infty, 0) \cup (0, \infty)$; no intercepts; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$ and $(0, \infty)$

11. $D = (-\infty, 0) \cup (0, \infty)$, $R = (-\infty, 0) \cup (0, \infty)$; no intercepts; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$ and $(0, \infty)$

12. $D = (-\infty, 0) \cup (0, \infty)$, $R = (0, \infty)$; no intercepts; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$; decreasing: $(0, \infty)$

13. $D = (-\infty, 0) \cup (0, \infty)$, $R = (-\infty, 0) \cup (0, \infty)$; no intercepts; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$ and $(0, \infty)$
Chapter 2 Answer Appendix
28. $D = (0, \infty), R = (-\infty, 0)$; no intercepts; $\lim_{x \to \infty} f(x) = 0$; continuous on $(0, \infty)$; increasing: $(0, \infty)$

29. $D = (-\infty, 0) \cup (0, \infty)$, $R = (0, \infty)$; no intercepts; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing: $(-\infty, 0)$; decreasing: $(0, \infty)$

34. $f(x) = 3\sqrt{6 + 3x}$

35. $g(x) = -2\sqrt{1024 + 8x}$

36. $f(x) = -\frac{2}{3}\sqrt{16x + 48} - 3$

37. $h(x) = 4 + \sqrt{7x - 12}$

38. $D = (-\infty, 0.25]$, $R = [-16, \infty)$; $x$-intercept: $-1.34$, $y$-intercept: $-15$; $\lim_{x \to \infty} f(x) = \infty$; continuous on $(-\infty, 0.25]$; decreasing: $(-\infty, 0.25)$

39. $D = (-\infty, \infty)$, $R = (-\infty, -49.00]$; $x$-intercept: $-52.66$, $y$-intercept: $-\infty$; $\lim_{x \to \infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$; continuous for all real numbers; increasing: $(-\infty, 0.28)$; decreasing: $(0.28, \infty)$

40. $D = (-\infty, \infty)$, $R = (-\infty, \infty)$; $x$-intercept: $-2034.5$, $y$-intercept: $-6.5$; $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$; continuous for all real numbers; decreasing: $(-\infty, \infty)$

41. $D = [1, 22)$, $R = [-\sqrt{63}, \sqrt{21}]$; $x$-intercept: $6.25$; continuous on $[1, 22)$; decreasing: $(1, 22)$

42a. Velocity of Water with a Nozzle

42b. $D = [0, \infty), R = [0, \infty)$; $\lim_{x \to \infty} f(x) = \infty$; continuous on $[0, \infty)$; increasing: $(0, \infty)$
80a. Sample answers given.

80c. Sample answer: When $0 < n < 1$, the average rate of change of the function decreases as $x$ approaches infinity. When $n = 1$, the average rate of change of the function is constant as $x$ approaches infinity. When $n > 1$, the average rate of change of the function increases as $x$ approaches infinity.

81. \[
\sqrt{\frac{8^0 \cdot 2^7}{4^{-n}}} = \sqrt{\frac{(2^3)^0 \cdot 2^7}{(2^2)^{-n}}} = \sqrt{\frac{2^{3n} \cdot 2^7}{2^{-2n}}} = \sqrt{2^{3n+7}} = \sqrt{2^{4n+6 \cdot 2^n+1}} = \sqrt{2^{4n+6 \cdot \sqrt{2^{n+1}}}} = 2^{2n+3 \sqrt{2^{n+1}}}
\]

82d. Sample answer: If the exponent is less than 0, the power is greater than 0 and less than 1. If the exponent is greater than 0 and less than 1, the power is greater than 1 and less than the base. If the exponent is greater than 1, the power is greater than the base. Any nonzero number to the zero power is 1. Thus, if the exponent is less than 0, the power is less than 1.

A power of a positive number is never negative, so the power is greater than 0. Any nonzero number to the zero power is 1 and to the first power is itself. Thus, if the exponent is greater than 0 and less than 1, the power is between 1 and the base. Any number to the first power is itself. Thus, if the exponent is greater than 1, the power is greater than the base.

85. Sample answer: As $n$ increases, the value of $\frac{1}{n}$ approaches 0.

This means that the value of $x^n$ will approach 1 when $x$ is positive and $-1$ when $x$ is negative. Therefore, for positive values of $x$, $f(x)$ will approach $1 + 5$ or 6 and will resemble the line $y = 6$. For negative values of $x$, $f(x)$ will approach $-1 + 5$ or 4 and will resemble the line $y = 4$. 

93. 

92. 

91. 

89. 

11.
### Page 96, Explore 2-2

3. Sample answer:

<table>
<thead>
<tr>
<th>Lead Coefficient</th>
<th>Degree of Polynomial</th>
<th>End Behavior</th>
</tr>
</thead>
</table>
| negative          | odd                  | \( \lim_{x \to -\infty} f(x) = -\infty \)
|                   |                      | \( \lim_{x \to \infty} f(x) = -\infty \) |
| negative          | even and nonzero     | \( \lim_{x \to -\infty} f(x) = -\infty \)
|                   |                      | \( \lim_{x \to \infty} f(x) = -\infty \) |
| positive          | odd                  | \( \lim_{x \to -\infty} f(x) = \infty \)
|                   |                      | \( \lim_{x \to \infty} f(x) = \infty \) |
| positive          | even and nonzero     | \( \lim_{x \to -\infty} f(x) = \infty \)
|                   |                      | \( \lim_{x \to \infty} f(x) = \infty \) |

### Page 102, Lesson 2-2 (Guided Practice)

6A. a. The degree is 5, and the leading coefficient is \(-54\), so \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \).

b. The distinct real zeros are \( x = 0 \), \( x = 4 \), and \( x = \frac{1}{3} \). The zero at \( x = \frac{1}{3} \) has multiplicity 3.

c. \[
\begin{array}{|c|c|c|c|}
\hline
\text{Interval} & \text{x-value} & f(x) & (x, f(x)) \\
\hline
(\infty, 0) & -1 & f(-1) = 640 & (-1, 640) \\
(0, \frac{1}{3}) & \frac{1}{4} & f\left(\frac{1}{4}\right) \approx -0.03 & \left(\frac{1}{4}, -0.03\right) \\
\left(\frac{1}{3}, 4\right) & 1 & f(1) = 48 & (1, 48) \\
(4, -\infty) & 5 & f(5) = -27,440 & (5, -27,440) \\
\hline
\end{array}
\]

d. \[ f(x) = -2x(x - 4)(3x - 1)^2 \]

6B. a. The degree is 3, and the leading coefficient is \(-1\), so \( \lim_{x \to -\infty} f(x) = \infty \) and \( \lim_{x \to \infty} f(x) = -\infty \).

b. The distinct real zeros are \( x = -2 \), \( x = 0 \), and \( x = 4 \). There is no multiplicity.

c. \[
\begin{array}{|c|c|c|c|}
\hline
\text{Interval} & \text{x-value} & f(x) & (x, f(x)) \\
\hline
(\infty, -2) & -4 & f(-4) = 64 & (-4, 64) \\
(-2, 0) & -1 & f(-1) = -5 & (-1, -5) \\
(0, 4) & 2 & f(2) = 16 & (2, 16) \\
(4, -\infty) & 10 & f(10) = -720 & (10, -720) \\
\hline
\end{array}
\]
20. The degree is 3, and the leading coefficient is $-1$. Because the degree is odd and the leading coefficient is negative, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty.
\]

21. The degree is 6, and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

22a. The degree is 4, and the leading coefficient is 3.87. Because the degree is even and the leading coefficient is positive, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

22b. The degree is 4, and leading coefficient is $43.77$. Because the degree is even and the leading coefficient is positive, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

23. 5 real zeros and 4 turning points; $0$, $-1$, and $-2$

24. 6 real zeros and 5 turning points; $0$, $6$, and $2$

25. 4 real zeros and 3 turning points; $\pm \sqrt{3}$

26. 4 real zeros and 3 turning points; $0$, $8$, and $-4$

27. 6 real zeros and 5 turning points; $2$ and $\sqrt{2}$

28. 8 real zeros and 7 turning points; no real zeros

29. 6 real zeros and 5 turning points; $0$, $-2$, and $2$

30. 5 real zeros and 4 turning points; $0$, $5$, and $-5$

31. 4 real zeros and 3 turning points; $0$, $\frac{1}{2}$, and $\frac{3}{2}$

32. 5 real zeros and 4 turning points; $0$, $\frac{1}{3}$, and $-5$

33a. The degree is 4, and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

33b. $0$, $-4$, $1$ (multiplicity: 2)

33c. Sample answer: $(-5, 180)$, $(-2, -36)$, $(0.5, 0.5625)$, $(2, 12)$

33d. $f(x) = x(x-3)(x-2)^2$

34a. The degree is 4, and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

34b. $0$ (multiplicity: 2), $4$, $-2$

34c. Sample answer: $(-3, 63)$, $(-1, -5)$, $(2, -32)$, $(5, 175)$

35a. The degree is 4, and the leading coefficient is $-1$. Because the degree is even and the leading coefficient is negative, 
\[
\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty.
\]

35b. $0$, $-3$ (multiplicity: 2), $5$

35c. Sample answer: $(-4, -36)$, $(-1, -24)$, $(2, 150)$, $(6, -486)$

35d. $f(x) = x^2(x-4)(x+2)$

36a. The degree is 4, and the leading coefficient is $2$. Because the degree is even and the leading coefficient is positive, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

36b. $0$, $-5$ (multiplicity: 2), $3$

36c. Sample answer: $(-6, 108)$, $(-1, 128)$, $(1, -144)$, $(4, 648)$

36d. $f(x) = -x(x+3)^2(x-5)$

37a. The degree is 5, and the leading coefficient is $-1$. Because the degree is odd and the leading coefficient is negative, 
\[
\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty.
\]

37b. $0$, $3$, $-2$ (multiplicity: 3)

37c. Sample answer: $(-3, 18)$, $(-1, -4)$, $(1, 54)$, $(4, -864)$

37d. $f(x) = -x(x-3)(x+2)^2$
38a. The degree is 4, and the leading coefficient is $-1$. Because the degree is even and the leading coefficient is negative, 
$$\lim_{x \to -\infty} f(x) = -\infty$$ \quad and \quad $$\lim_{x \to \infty} f(x) = -\infty.$$
38b. $-2$ (multiplicity: 2), 4 (multiplicity: 2)
38c. Sample answer: $(-3, -49), (-1, -25), (5, -49)$
38d.

39a. The degree is 3, and the leading coefficient is 3. Because the degree is odd and the leading coefficient is positive, 
$$\lim_{x \to -\infty} f(x) = -\infty$$ \quad and \quad $$\lim_{x \to \infty} f(x) = \infty.$$
39b. 0, 4, -3
39c. Sample answer: $(-4, -96), (-2, 36), (2, -60), (5, 120)$
39d.

40a. The degree is 3, and the leading coefficient is $-2$. Because the degree is odd and the leading coefficient is negative, 
$$\lim_{x \to -\infty} f(x) = \infty$$ \quad and \quad $$\lim_{x \to \infty} f(x) = -\infty.$$
40b. 0, -3, 1
40c. Sample answer: $(-4, 40), (-2, -12), (0.5, 1.75), (2, -20)$
40d.

41a. The degree is 4, and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, 
$$\lim_{x \to -\infty} f(x) = \infty$$ \quad and \quad $$\lim_{x \to \infty} f(x) = \infty.$$
41b. 0 (multiplicity: 2), 4, -5
41c. Sample answer: $(-6, 360), (-2, -72), (2, -56), (5, 250)$
41d.

42a. The degree is 5, and the leading coefficient is 1. Because the degree is odd and the leading coefficient is positive, 
$$\lim_{x \to -\infty} f(x) = -\infty$$ \quad and \quad $$\lim_{x \to \infty} f(x) = \infty.$$
42b. 0 (multiplicity: 3), -5, 2
42c. Sample answer: $(-6, -1728), (-3, 270), (1, -6), (3, 216)$
42d.

44. Sample answer: $f(x) = -1.25x + 5$
45. Sample answer: $f(x) = 0.09x^3 - 2.70x^2 + 24.63x - 65.21$
46. Sample answer: $f(x) = 0.88x^4 + 0.89x^3 - 1.71x^2 - 2.99x + 4.89$
47. Sample answer: $f(x) = 4.05x^4 - 0.09x^3 + 6.69x^2 - 222.03x + 2687.74$
49a. According to the data, the values are increasing as $x$ increases, therefore, 
$$\lim_{x \to \infty} f(x) = \infty.$$
49b. \[
\begin{array}{c}
\text{Sample answer: } f(x) = \frac{0.146x^4 - 3.526x^3 + 32.406x^2 - 63.374x + 473.255}{200} \\
The line is not a good fit. There are many outlying points.
\end{array}
\]
49c. The leading coefficient is 0.146, so 
$$\lim_{x \to \infty} f(x) = \infty.$$ Sample answer: The prediction was accurate because the leading coefficient is positive, as $x \to \infty, f(x) \to \infty.$
68b. \[
A(r) = 2\pi r^2 + \frac{30}{r}
\]
68c. \[
\text{about 33.7 } \ln^2
\]
69. $f(x) = x^3 - 8x^2 - 3x + 90$
70. $f(x) = x^3 + 6x^2 - 24x - 64$
71. $f(x) = x^5 - 9x^4 + 23x^3 - 3x^2 - 36x$
72. $f(x) = x^5 - 4x^4 - 19x^3 + 46x^2 - 24x$
73. $f(x) = 12x^4 + 83x^3 + 131x^2 - 54x - 72$
74. $f(x) = 6x^5 - 23x^4 - 39x^3 + 15x^2 + 25x$
76. Sample answer:
\[ f(x) = (x - 2)(x - 5) \cdot (x + 1)(x^2 + 4) \]

77. Sample answer:
\[ f(x) = -(x + 2) \cdot (x + 3)(x - 4) \cdot (x - 1)(x^2 + 6) \]

78. Sample answer:
\[ f(x) = (x + 1) \cdot (x + 2)^2(x^2 + 1) \]

79. Sample answer:
\[ f(x) = -(x - 1) \cdot (x - 2)(x + 1)^2(x^2 + 2) \]

80a. Sample answer: According to the correlation coefficients for each of the models, \( r^2 = 0.94 \) for the cubic function and \( r^2 = 0.96 \) for the quartic function. Therefore, the quartic function is a better model.

80b. \( f(x) = -0.009x^3 - 0.230x^2 + 2.305x + 3.796 \)

80c. \( f(x) = 0.012x^4 - 0.225x^3 + 0.978x^2 + 0.152x + 4.312 \)

89a. Sample answer: 
\[ f(x) = a(x - 1)(x + 4)(x - 5) \]
\[ g(x) = 5a(x - 1)(x + 4) \cdot (x - 5) \]

89b. Sample answer: Very close to the origin, the function approximates the behavior of the term of lower degree, or \( h(x) \).

89c. Sample answer: As \( x \to \infty \) and \( -\infty \), the function approximates the behavior of the term of higher degree, or \( g(x) \).

89d. Sample answer: As \( x \to \infty \) and \( -\infty \), the function approximates the behavior of function \( a \) and, very close to the origin, the function approximates the behavior of function \( b \).

90. Colleen; sample answer: The data suggest that there are two turning points. Thus, a cubic function is a better representation than a quadratic function. Also, when the table is entered into a graphing calculator, \( r^2 = 0.99 \) for the cubic function and \( r^2 = 0.75 \) for the quadratic function. This further supports Colleen’s model.

91. Sample answer: No; a polynomial function \( f(x) \) cannot have both an absolute maximum and absolute minimum because as \( x \to \infty \), \( f(x) \) will approach either \( \infty \) or \( -\infty \). If \( f(x) \to \infty \) as \( x \to \infty \), then an absolute maximum is impossible. If \( f(x) \to -\infty \) as \( x \to -\infty \), then an absolute minimum is impossible.

92. Sample answer: The function \( f(x) = 0 \) has no degree because there are no terms for the polynomial. The function \( g(x) = c \), \( c \neq 0 \), has degree 0 because \( c = cx^0 \) for all \( x \).

93. Rearranging the terms gives \( f(x) = x^3 - x^2 - 12x + 5x^2 - 5x - 60 \). Notice how the first set of three terms has a common factor of \( x \) and the second set of three terms has a common factor of 5. After factoring using the Distributive Property, \( f(x) = x(x^2 - x - 12) + 5(x^2 - x - 12) \). Now notice how the factors inside the parentheses are identical. Using the Commutative Property, \( f(x) = (x + 5)(x^2 - x - 12) \). After factoring the second factor, \( f(x) = (x + 5)(x - 4)(x + 3) \). The function is now completely factored and the zeros of the function are \(-5, 4, \) and \(-3\). These were determined by setting each factor equal to zero and solving for \( x \).

94. Sample answer: This is possible because one of the graphs could be more compressed than the other. For example, \( f(x) = a(x - 1)(x + 4)(x - 5) \) and \( g(x) = 5a(x - 1)(x + 4) \cdot (x - 5) \) have the same zeros, degree, and end behavior, assuming \( a > 0 \). The graph of \( g(x) \) will be more stretched due to the factor of 5.
95. Sample answer: There is one turning point at the absolute maximum, one at the relative minimum, and one at the relative maximum. Therefore, the minimum degree is 3 + 1 or 4.

96. Sample answer: Make a scatter plot of the data. Use the scatter plot to determine which degree polynomial most resembles the data. Find the regression equation for the polynomial and compare the absolute value of its correlation coefficient to 1. Graph this equation on the same screen as the scatter plot to confirm their similarity. If the model does not fit the data or the absolute value of the correlation coefficient is not close to 1, correlation coefficients for other polynomials can be found to see if there is a better fit.

104. The graph of \( g(x) \) is the graph of \( f(x) \) translated 4 units right and 3 units up; \( g(x) = (x - 4)^2 + 3 \).

105. The graph of \( g(x) \) is the graph of \( f(x) \) reflected in the x-axis and translated 2 units left and 4 units down; \( g(x) = -(x + 2)^2 - 4 \).

106. The graph of \( g(x) \) is the graph of \( f(x) \) translated 6 units right and 4 units down; \( g(x) = (x - 6)^2 - 4 \).

Page 116, Lesson 2-3

62. Sample answer: I would use my graphing calculator to graph the polynomial and to determine the three integral zeros, \( a, b, \) and \( c \). I would then use synthetic division to divide the polynomial by \( a \). I would then divide the resulting depressed polynomial by \( b \), and then the new depressed polynomial by \( c \). The third depressed polynomial will have degree 2. Finally, I would factor the second-degree polynomial to find the two non-integral, rational zeros, \( d \) and \( e \). So, the polynomial is either the product \((x - a)(x - b)(x - c)(x - d)(x - e)\) or this product multiplied by some rational number.

68. Sample answer: Both long division and synthetic division can be used to divide a polynomial by a linear factor. Long division can also be used to divide a polynomial by a nonlinear factor. In synthetic division, only the coefficients are used. In both long division and synthetic division, placeholders are needed if a power of a variable is missing.

Page 118, Mid-Chapter Quiz

1. The graph of \( f(x) = 2x^3 - 8x \);
\[
\begin{align*}
D &= (-\infty, \infty), R = (-\infty, \infty); \\
\text{intercept: } 0, \quad &\lim_{x \to -\infty} f(x) = -\infty \\
&\text{and } \lim_{x \to \infty} f(x) = \infty; \text{ continuous for all real numbers; increasing: } (-\infty, \infty)
\end{align*}
\]

2. The graph of \( f(x) = -\frac{3}{2}x^2 \);
\[
\begin{align*}
D &= (-\infty, \infty), R = (-\infty, 0]; \quad &\text{intercept: } 0, \quad &\lim_{x \to -\infty} f(x) = -\infty \quad \text{and } \lim_{x \to \infty} f(x) = -\infty; \text{ continuous for all real numbers; increasing: } (-\infty, 0); \text{ decreasing: } (0, \infty)
\end{align*}
\]
51. $-3, -2, 1, 3 + i, 3 - i; g(x) = (x + 3)(x + 2)(x - 1) \cdot (x - 3 + i)(x - 3 - i)$

52. $5, -\frac{3}{4}, 2, 4 + i, 4 - i; g(x) = (x - 5)(4x + 3)(x - 2) \cdot (x - 4 - i)(x - 4 + i)$

53. $3, 2\sqrt{2}, -2\sqrt{2}, 2i, -2i; f(x) = (x - 3)(x + 2\sqrt{2}) \cdot (x - 2\sqrt{2})(x + 2i)(x - 2i)$

54. $2 + \sqrt{3}, 2 - \sqrt{3}, 3 + i, 3 - i; g(x) = (x - 2 - \sqrt{3}) \cdot (x - 2 + \sqrt{3})(x - 3 + i)(x - 3 + i)$

72b. i. $3i, -3i, 1$
   ii. $-\frac{1}{2}, -2i, 2i, -2, 2$
   iii. $-2, \frac{3}{5}, 1$
   iv. $-3i, 3i, -4i, 4i$
   v. $4i, -4i, -2, 0$ (multiplicity: 2), $\frac{1}{3}$
   vi. $i, -i, \frac{3}{4}, 2$

72c. An odd-degree polynomial function always has an odd number of zeros, and a polynomial function with real coefficients has imaginary zeros that occur in conjugate pairs. Therefore, an odd function with real coefficients will always have at least one real zero.

75a. The graphs of $-f(x)$ are the graphs of $f(x)$ reflected in the $x$-axis. The zeros are the same.

75b. The graphs of $f(-x)$ are the graphs of $f(x)$ reflected in the $y$-axis. The zeros are opposites.

77. False; sample answer: The third-degree polynomial $x^3 + x^2 - 17x + 15$ has three real zeros and no nonreal zeros.

82. Sample answer: $c$ must be greater than or equal to $\frac{a_2}{a_1}$. If $c$ is less than $\frac{a_2}{a_1}$, then the second term of the depressed polynomial will be negative and the upper bound test will fail.

83. Sample answer: If a polynomial has an imaginary zero, then its complex conjugate is also a zero of the polynomial. Therefore, any polynomial that has one imaginary zero has at least two imaginary zeros.

Pages 138–140, Lesson 2-5

9. asymptotes: $x = -4, x = 5, y = 1$;
   $x$-intercepts: $-2, 3$;
   $y$-intercept: $\frac{3}{10}$;
   $D = \{x \mid x \neq -4, 5, x \in \mathbb{R}\}$

10. asymptotes: $x = 1, x = -2, y = 2$;
   $x$-intercepts: $\frac{3}{2}$;
   $6; y$-intercept: 9;
   $D = \{x \mid x \neq -1, -2, x \in \mathbb{R}\}$

11. asymptotes: $x = -2, x = 2, y = 0$;
   $x$-intercept: $-2$;
   $D = \{x \mid x \neq -2, 2, x \in \mathbb{R}\}$

12. asymptotes: $x = 0, x = 6, y = 0$;
   $x$-intercept: $-2$;
   $D = \{x \mid x \neq 0, 6, x \in \mathbb{R}\}$

13. asymptotes: $x = -5, x = 6, y = 0$;
   $x$-intercept: $-2$;
   $y$-intercept: $\frac{1}{15}$;
   $D = \{x \mid x \neq -5, 6, x \in \mathbb{R}\}$

14. asymptotes: $x = 0, x = 5, y = -2$;
   $y = 0$; $x$-intercepts: $-6, -4$;
   $D = \{x \mid x \neq -2, 0, 5, x \in \mathbb{R}\}$

15. asymptotes: $x = -3, x = -1$; $x$-intercepts: $-5, 2, 0$; $y$-intercept: 0;
   $D = \{x \mid x \neq -3, -1, x \in \mathbb{R}\}$
16. asymptotes: $x = -3$, $x = 8$; x-intercepts: $-6, 0, 4$; y-intercept: 0; $D = \{x \mid x \neq -3, 8, x \in \mathbb{R}\}$

17. asymptote: $y = 0$; x-intercept: 8; y-intercept: $-\frac{8}{5}$; $D = \{x \mid x \in \mathbb{R}\}$

18. asymptote: $y = 0$; y-intercept: $-\frac{2}{3}$; $D = \{x \mid x \in \mathbb{R}\}$

19c. **Car Wash**

<table>
<thead>
<tr>
<th>Week</th>
<th>p(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
</tr>
</tbody>
</table>

$p(x) = \frac{3x^2 - 3}{2x^2 + 7x + 5}$

20. asymptotes: $x = 0$, $y = 0$; x-intercept: $\frac{4}{3}$; $D = \{x \mid x \neq 0, x \in \mathbb{R}\}$

21. asymptotes $x = \frac{1}{4}$; $y = 0$; y-intercept: $\frac{1}{4}$; $D = \{x \mid x \neq \sqrt{\frac{4}{3}}, x \in \mathbb{R}\}$

22. asymptotes: $y = 1$; $x = -3, x = -1$; x-intercepts: 3, 5; y-intercept: -5; $D = \{x \mid x \neq -3, -1, x \in \mathbb{R}\}$

23. asymptotes: $x = 4$, $y = 1$; x-intercept: -7; y-intercept: $-\frac{7}{4}$; $D = \{x \mid x \neq 4, x \in \mathbb{R}\}$

24. asymptote: $x = -3$; x-intercept: 0; y-intercept: 0; $D = \{x \mid x \neq -3, x \in \mathbb{R}\}$

25. asymptotes: $x = 4$; x-intercepts: -2, -1, 0; y-intercept: 0; $D = \{x \mid x \neq 4, x \in \mathbb{R}\}$

26. asymptotes: $x = -1, x = 2$; $y = 0$; hole: (-3, -1); x-intercept: 7; y-intercept: $\frac{2}{7}$; $D = \{x \mid x \neq -3, -1, 2, x \in \mathbb{R}\}$

27. asymptotes: $x = -1$, $y = 0$; holes: (-2, -1), $(2, \frac{1}{3})$; y-intercept: 1; $D = \{x \mid x \neq -2, -1, 2, x \in \mathbb{R}\}$
28. asymptotes: \(x = -3;\) 
   \(y = 1;\) hole: \(1, \frac{3}{4}\) 
   
   \(x\)-intercept: \(-4;\) 
   \(y\)-intercept: \(\frac{4}{3};\) 
   
   \(D = \{ x | x \neq -3, 1, \} \quad x \in \mathbb{R}\) 

29. asymptotes: \(x = -4;\) 
   \(y = 0;\) hole: \(\frac{1}{81};\) 
   \(x\)-intercept: \(-1, 2;\) 
   \(y\)-intercept: \(\frac{1}{16};\) 
   
   \(D = \{ x | x \neq -4, 5, \} \quad x \in \mathbb{R}\) 

42a. 
   ![Graph](image1) 
   
   [0, 100] scl: 10 by [0, 10,000] scl: 1000 

42b. 
   ![Graph](image2) 
   
   about 88.9% 

42c. No; sample answer: The function is not defined when \(x = 100.\) This suggests that it is not financially feasible to remove 100\% of the salt at the plant.

45a. There is a vertical asymptote at \(r_1 = 30\) and a horizontal asymptote at \(r_2 = 30.\) 

45c. No; sample answer: As \(r_1\) approaches infinity, \(r_2\) approaches 30. This suggests that the average speeds reached during the first leg of the trip have no bounds. Being able to reach an infinite speed is not reasonable. The same holds true about \(r_2\) as \(r_1\) approaches 30 from the right.

52a. Sample answer: The concentration of the total solution is the sum of the amount of acetic acid in the original 10 liters and the amount in the \(a\) liters of the 60\% solution, divided by the total amount of solution or \(\frac{0.60a + 0.20(10)}{a + 10}.\) Multiplying both the numerator and the denominator by \(5\) gives \(\frac{3a + 10}{5a + 50}\) 

52b relevant domain: real numbers \(a\) such that \(0 \leq a \leq 90;\) horizontal asymptote: \(y = 0.6\) 

52c. Sample answer: Because the tank already has 10 liters of solution in it and it will only hold a total of 100 liters, the amount of solution added must be less than or equal to 90 liters. It is also impossible to add negative amounts of solution, so the amount added must be greater than or equal to 0. As you add more of the 60\% solution, the concentration of the total solution will get closer to 60\%, but because the solution already in the tank has a lower concentration, the concentration of the total solution can never reach 60\%. Therefore, there is a horizontal asymptote at \(y = 0.6.\) 

52d. Yes; sample answer: The function is not defined at \(a = -10,\) but because the value is not in the relevant domain, the asymptote does not pertain to the function. If there were no domain restrictions, there would be a vertical asymptote at \(a = -10.\) 

53b. 
   ![Graph](image3) 
   
   [−5, 10] scl: 1 by [−1, 1] scl: 0.1 

53d. Sample answer: When the degree of the numerator is less than the degree of the denominator and the numerator has at least one real zero, the graph of the function will have \(y = 0\) as an asymptote and will intersect the asymptote at the real zeros of the numerator.

54. Sometimes; sample answer: When \(a = d,\) the function will have a horizontal asymptote at \(y = 1.\) When \(a \neq d,\) the function will not have a horizontal asymptote at \(y = 1.\) 

59. Sample answer: The test intervals are used to determine the location of points on the graph. Because many rational functions are not continuous, one interval may include \(y\)-values that are vastly different than the next interval. Therefore, at least one, and preferably more than one, point is needed for every interval in order to sketch a reasonably accurate graph of the function.
Pages 145–147, Lesson 2-6

62b. \( f(x) = \frac{x - 1}{x + 2} \)  
\([-10, 10]\) scl: 1 by \([-10, 10]\) scl: 1

\( g(x) = \frac{2x - 5}{x - 3} \)  
\([-10, 10]\) scl: 1 by \([-10, 10]\) scl: 1

\( h(x) = \frac{x + 4}{3x - 1} \)  
\([-10, 10]\) scl: 1 by \([-10, 10]\) scl: 1

62c. i. \( f(x) = \frac{x - 1}{x + 2} \)

\( x \)  
\(-2\)  
Und. 0 (+)

Negative  Negative  Positive

ii. \( g(x) = \frac{2x - 5}{x - 3} \)

\( x \)  
\( 0 \)  
Und. 0 (+)

Negative  Negative  Positive

iii. \( h(x) = \frac{x + 4}{3x - 1} \)

\( x \)  
\(-4\)  
Und. 0 (+)

Positive  Positive  Positive

62d. i. \( f(x) \leq 0 \) on \((-\infty, -2) \cup (-2, 1)\)

ii. \( g(x) \geq 0 \) on \([\frac{2}{5}, 3]\) and \((-\infty, -4) \cup (-4, \frac{1}{3}) \cup (\frac{1}{3}, \infty)\)

iii. \( h(x) > 0 \) on \((-\infty, -4) \cup (-4, \frac{1}{3}) \cup (\frac{1}{3}, \infty)\)

70. \( D = \{x | x \neq -4, x \in \mathbb{R}\}; x = -4; y = 2\)

71. \( D = \{x | x \neq -6, x \in \mathbb{R}\}; x = -6\)

72. \( D = \{x | x \neq \frac{1}{2}, 5, x \in \mathbb{R}\}; x = \frac{1}{2}; x = 5, y = 0\)

78a. \( f(x) = 0.003x^2 - 0.111x^2 + 0.019x + 30.259 \)

78b. about \$8.42

78c. \( 1, 2; 3, 4 \) scl: 5 by \([0, 35]\) scl: 5

Page 149, Study Guide and Review

81a. \( f(t) \) is a quadratic polynomial function. \( g(t) \) is a power or radical function.

81b. Sample answer: Since time and the amount of water cannot be negative, the relevant domains for \( f(t) \) and \( g(t) \) are restricted to nonnegative values for \( t \) result in nonnegative values for \( f(t) \) and \( g(t) \). For \( f(t) \), \( D = [0, 8.89] \) and \( R = [0, 86.25] \). For \( g(t) \), \( D = [0, \infty) \) and \( R = [0, \infty) \).

81c. \( \lim_{t \to 8.89} f(t) = 0; \lim_{t \to 0} f(t) = 80; \lim_{t \to 0} g(t) = 0 \)

81d. \( \lim_{t \to \infty} g(t) = \infty \)

81e. Sample answer: \( f(x) \) is a polynomial function, so it is also continuous and the Intermediate Value Theorem applies. Therefore, since \( f(3) = 74 \) and \( f(5) = 30 \), it follows that there is a number \( c \), such that \( 3 < c < 5 \) and \( f(c) = 50 \).

81f. about 5.89; Sample answer: This means that the town will run out of water reserves after about 5.89 years.

81g. about 5.23 years

11. \( f(x) = 5x^2 \)

D = \((-\infty, \infty)\), \( R = [0, \infty) \); intercept: 0; \( \lim_{x \to \infty} f(x) = \infty \)

and \( \lim_{x \to 0} f(x) = \infty \); continuous for all real numbers; decreasing: \((-\infty, 0)\); increasing: \((0, \infty)\)

12. \( f(x) = -8x^3 \)

D = \((-\infty, \infty)\), \( R = (-\infty, \infty) \); intercept: 0; \( \lim_{x \to \infty} f(x) = \infty \)

and \( \lim_{x \to 0} f(x) = -\infty \); continuous for all real numbers; decreasing: \((-\infty, \infty)\)

13. \( f(x) = x^3 \)

D = \((-\infty, 0) \cup (0, \infty)\), \( R = (-\infty, 0) \cup (0, \infty) \); no intercepts; \( \lim_{x \to \infty} f(x) = 0 \)

and \( \lim_{x \to 0} f(x) = 0 \); infinite discontinuity at \( x = 0 \); decreasing: \((-\infty, 0)\) and \((0, \infty)\)
Page 153, Practice Test

15a. The degree is 3 and the leading coefficient is 1. Because the degree is odd and the leading coefficient is positive, 
\[ \lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty. \]

15b. 0, 1, -3

15c. Sample answer: (-1, 4), (-2, 6), (2, 10), (3, 36)

15d. 

16a. The degree is 4 and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, 
\[ \lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty. \]

16b. -3, 3, 0 (multiplicity 2)

16c. Sample answer: (-1, -8), (1, -8), (2, -20), (4, 112)

16d. 

19a. 

Average High Temperature

19b. \[ f(x) = -0.071x^3 + 0.415x^2 + 5.909x + 54.646 \]

19c. 44.9°

25. 

asymptotes: \( x = -5, y = 2 \); 
\( x \)-intercept: 3; \( y \)-intercept: \( -\frac{6}{5} \); 
\( D = \{ x \mid x \neq -5, x \in \mathbb{R} \} \)

26. 

asymptotes: \( x = 4, y = x + 5 \); 
\( x \)-intercepts: -3 and 2; 
\( y \)-intercept: \( \frac{3}{2} \); 
\( D = \{ x \mid x \neq 4, x \in \mathbb{R} \} \)